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Compactness of Isospectral Sets

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Let M be a compact Riemannian manifold, and $\Delta = -\text{div}(\text{grad})$ the Laplacian on M . The solutions to the equation

$$\Delta(f) = \lambda \cdot f$$

are called eigenfunctions, and the corresponding λ 's are eigenvalues. It is not hard to see that there are countably many λ 's, tending to ∞ , and for each λ a finite-dimensional family of functions f , for which this equation has a solution. The λ 's may be thought of as the "fundamental frequencies" of M , just as a musical instrument has a series of overtones which are the frequencies at which it vibrates.

A fundamental question is: to what extent do the eigenvalues of M determine the geometry and topology of M ? This was framed by Mark Kac in 1966 as the question: Can one hear the shape of a drum?

Our first observation is that the general answer to the question is "no." Indeed, one has, by a construction of Sunada [Su], a powerful technique for constructing many different types of isospectral manifolds.

On the other hand, it should be the case that the spectrum of Δ tells us a lot about M . For instance, it has been well-known for many years that the spectrum of Δ of a surface determines the genus of the surface, and recent work of Osgood, Phillips, and Sarnak [OPS] shows that the spectrum of Δ of a surface determines the surface up to a compact family of metrics.

Our first result is:

Theorem 1 ([BPP]) *(a) Sets of isospectral manifolds of negative curvature contain only finitely many topological types.*

(b) *Sets of isospectral manifolds whose sectional curvatures are bounded from below contain only finitely many topological types.*

Theorem 2 ([BPP]) (a) *Sets of isospectral 3-manifolds of negative curvature are compact.*

(b) *Sets of isospectral 3-manifolds whose Ricci curvatures are bounded from below are compact.*

The fundamental idea behind these theorems is that if one can decompose a manifold into some fixed number of pieces, all of which are pretty much standard, then one can determine M by the finitely many ways in which these pieces can be assembled. This is an important idea behind the Cheeger Finiteness Theorem [Ch].

In part (a) in the theorems above, the finitely many pieces are given by balls of radius less than the injectivity radius of M , which are topologically disks. Once one has a bound on the injectivity radius, which one obtains from the asymptotics of the wave equation, one obtains the desired decomposition of M .

In part 1(b), the finitely many pieces are given by balls of a fixed radius, which may be topologically non-trivial, and may also be of very small volume. A theorem of Cheng [Cg] shows how to control the number of such pieces by the low eigenvalues of M , while an argument of Grove and Petersen [GP] shows that, even though these balls may be topologically non-trivial, their contribution to the topology of M can be controlled.

To obtain 2, we make use of the decomposition of M into bits to estimate the Sobolev constant of M . Knowing this, and using an estimate of Gilkey [Gi] on the leading terms in the asymptotics of the heat equation, one can get better and better bounds on the curvature and its covariant derivatives, thus establishing the desired compactness.

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