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Notes on Thermodynamic formalism for Anosov flows

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0. Anosov flows

Let M be a compact C^∞ Riemannian manifold and let $\phi_t : M \longrightarrow M$ be a $C^k (k \geqslant 1)$ flow, $t \in \mathbf{R}$.

DEFINITION. — We say that $\phi_t : M \longrightarrow M$ is *Anosov* if there is a continuous splitting of the tangent space $TM = E^0 \oplus E^s \oplus E^u$ into $D\phi_t$-invariant sub-bundles such that :

(i) E^0 is one-dimensional and tangent to the orbits of the flow ϕ;

(ii) There exist constants $C, \lambda > 0$ such that

$$\|D\phi_t|_{E^s}\| \leqslant C \cdot e^{-\lambda t} \text{ and } \|D\phi_{-t}|_{E^u}\| \leqslant C \cdot e^{-\lambda t}, \text{ for } t \geqslant 0$$

In addition, we shall always ask that the flow be *transitive* (*i.e.* there exists a dense ϕ-orbit in M) and *weak-mixing* (*i.e.* the least periods of closed orbits are not contained in a $\mathbf{Z}$, for some $a > 0$).

Remark. — The *hyperbolicity* of the flow can be seen through the induced action on the Banach space C^0 sections $\alpha \in \Gamma(M, TM)$ given by $(\phi_t^* \alpha)(x) = D\phi_t \alpha(\phi_{-t} x)$ and the following characterisation.

THEOREM (Mather). — *The spectrum of ϕ_t^* is contained in the union of : (a) the unit circle; (b) an annulus with radii $R_1 > R_2 > 1$; and (c) an annulus with radii $r_1 < r_2 < 1$.*

Principal example. — Let V be a compact n-dimensional manifold with negative sectional curvatures. Let $M = SV$ be the sphere bundle over V and let $\phi_t : M \longrightarrow M$ be the associated *geodesic flow.*

THEOREM (Anosov, 1967). — *The geodesic flow is Anosov with :*

$$(\operatorname{Dim} M = 2n - 1, \ \operatorname{Dim} E^u = n - 1 \textit{ and } \operatorname{Dim} E^s = n - 1)$$

The space of C^k all flows has a natural topology coming from the space of vector fields $\mathfrak{X}(M)$ on the manifold M. The stability of the definition is given by :

THEOREM (Anosov-Sinai, 1967).

(i) The Anosov flows are an open subset of all C^k flows on M;

(ii) The Anosov flows are structurally stable (i.e. for each Anosov ϕ and ψ sufficiently close there exists a homeomorphism $h = h_\psi : M \longrightarrow M$ such that h preserves the orbits-up to parameterisation).

(The proof of structural stability uses the implicit function theorem to find a solution $\psi \mapsto h_\psi$ in $C^0(M,M)$. To apply this requires Mather's theorem).

1. Closed orbits and Livsic's theorem

We say that a ϕ-orbit τ is closed if there exists $T>0$ such that $\phi_T x = x$ for all $x \in \tau$. The least such T is the least period $\lambda(\tau)$ of the closed orbit τ.

PROPOSITION. — *An Anosov flow has infinitely many closed orbits, and their union is dense in M.*

(This follows from *"Anosov Closing Lemma"* : If the ends of an orbit segment are sufficiently close it can be "closed-up" to give a closed orbit τ arbitrarily close by).

A function $F : M \longrightarrow \mathbf{R}$ is Hölder continuous if $|F(x) - F(y)| \leqslant$ Const. $\big(d(x,y)\big)^\alpha$, for some $\alpha > 0$. Given a continuous function $F : M \longrightarrow \mathbf{R}$ we denote $\lambda_F(\tau) = \int_0^{\lambda(\tau)} F(\phi_t x)dt$. The density of the closed orbits leads to :

THEOREM (Livsic). — *For Hölder continuous functions $F, G : M \longrightarrow \mathbf{R}$ the following are equivalent :*

(i) $\lambda_F(\tau) = \lambda_G(\tau)$, *for all closed orbits* τ;

(ii) There exists a Hölder continuous function $k : M \longrightarrow \mathbf{R}$ (differentiable in the flow direction) such that $F(x) = G(x) + \frac{dk(\varphi_t(x))}{dt}|_t = 0$. Furthermore, if $F, G \in C^\infty(M)$ then $k \in C^\infty(M)$ (Marco, Moriyon, de la Llave).

(*(ii)* $\Rightarrow$ *(i)* is easy. To construct $k : M \longrightarrow \mathbf{R}$ given condition *(i)* choose a dense ϕ-orbit γ and a starting point $x \in \gamma$. For $\phi_t x \in \gamma$ we define $k(\phi_t x) = \int_0^t (F-G)(\phi_u x)\cdot du$. The Anosov closing lemma and condition *(i)* imply $k|\gamma$ is Hölder. The definition of k extends to M by continuity).

Application. — If ϕ has a smooth (*i.e.* a.c. w.r.t. volume) invariant measure on M then the density is C^∞.

2. Topological entropy and the maximal measure

THEOREM (Margulis). — *There exists a constant $h > 0$ such that the least periods are distributed according to*

$$\frac{\#\{\tau|\lambda(\tau) \leqslant x\}}{e^{hx}/hx} \longrightarrow 1, \text{ as } x \longrightarrow +\infty$$

(Margulis proved this by introducing transverse measures for the horocycle foliations which scales by $e^{\pm ht}$ under ϕ_t^* and using a delicate application of the Anosov closing lemma).

DEFINITION. — The value $h = h(\phi) = \lim_{x \to \infty} \frac{\text{Log}\#\{\tau|\lambda(\tau) \leqslant x\}}{x}$ is called the *topological entropy* of the flow ϕ. Given any ϕ-invariant probability measure ν we denote the entropy of the time-one map $\phi_{t=1} : M \longrightarrow M$ by $h(\phi,\nu) \geqslant 0$.

For example :

$$\begin{cases} \text{for } \nu \text{ ergodic and } a.\ a.\ x \in M\ (w.r.t.\ \nu) \\ h(\phi,\nu) = \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{-1}{T} \text{Log}\, \nu\{y \in M | d(\phi_t x, \phi_t y) \leqslant \varepsilon\}. \end{cases}$$

THEOREM (Variational Principle). — *$h(\phi) \geqslant h(\phi,\nu)$ for all ϕ-invariant probability measures ν, and there is a (unique) ϕ-invariant probability measure μ such that $h(\phi) = h(\phi,\mu)$.*

DEFINITION. — The measure μ is called the *maximal measure* for the flow ϕ.

The Margulis result suggests that μ can be constructed from closed orbits :

THEOREM (Bowen). — *For any continuous function $F \in C^0(M)$:*

$$\frac{\sum_{\lambda(\tau) \leqslant x} \frac{\lambda_F(\tau)}{\lambda(\tau)}}{\#\{\tau|\lambda(\tau) \leqslant x\}} \longrightarrow \int F d\mu, \quad \textit{as } x \longrightarrow +\infty.$$

3. Equilibrium states and the SRB-measure

Let $K : M \longrightarrow \mathbf{R}$ be a Hölder continuous function.

PROPOSITION (Variational Principle). — *There exists a (unique) ϕ-invariant probability measure μ such that $h(\phi,\mu) + \int K d\mu \geqslant h(\phi,\nu) + \int K d\nu$ for all ϕ-invariant probability measures ν.*

DEFINITIONS. — The measure μ is called the *equilibrium state* for K and the real number $P(K,\phi) := h(\phi,\mu) + \int K d\mu$ the *pressure* of the function K.

PROPOSITION (Livsic-Sinai). — *If two Hölder functions K_1, K_2 have the same equilibrium state then*

$$K_1(x) = K_2(x) + \frac{dk(\phi_t x)}{dt}|_t = 0 + \textit{ constant },$$
$$\textit{for some } k : M \longrightarrow \mathbf{R}.$$

If we take $K \equiv 0$ then $P(K) = h(\varphi)$ and the equilibrium state μ is the maximal measure. A more interesting case : $K(x) = \lambda^u(x) := \lim_{t\to\infty} \frac{\mathrm{Log}\,|\mathrm{Det}(D\phi_t|_{E^u})|}{t}$ (i.e. the infinitesimal expansion in the unstable direction).

DEFINITION. — The equilibrium state m for λ^u is called the *Sinai-Ruelle-Bowen measure* (or SRB-measure, for short).

THEOREM (Sinai).

(i) For almost all points $x \in M$ (w.r.t. the Riemannian measure on M) and any $F \in C^0(M)$, $\frac{1}{t}\int_0^t F(\varphi_u(x))du \longrightarrow \int F dm$, as $t \longrightarrow \infty$.

(ii) If ϕ has a smooth invariant probability measure then it is precisely the SRB measure.

(Part (ii) follows from part (i) by the Birkhoff ergodic theorem).

4. Reducing to an expanding map ("Symbolic Dynamics")

We describe the rough idea of a standard approach is to reduce the study of $\phi_t : M \longrightarrow M$ to that of an associated expanding map.

Step 1 : (Markov Poincaré sections). — We choose a finite family $\{T_1, \ldots, T_k\}$ of (small) disjoint codimension-one sections transverse to the flow. We then define

$$\begin{cases} r(x) = \inf\{t > 0 | \phi_t(x) \in \bigcup_i T_i\} & \text{(Poincaré return time)} \\ P(x) = \phi_{t=r(x)}(x) & \text{(Poincaré return map)} \end{cases}$$

Extra hypothesis. — It can be arranged that for each $i = 1, \ldots, k$ the sets

$$\begin{cases} W^s(x,T_i) = \{y \in T_i | d(P^i x, P^i y) \to 0, & \text{as } i \to \infty \text{ and } d(P^i x, P^i y) \leqslant \varepsilon, \forall i \geqslant 0\} \\ W^u(x,T_i) = \{z \in T_i | d(P^{-i}x, P^{-i}z) \to 0, & \text{as } i \to \infty \text{ and } d(P^{-i}x, P^{-i}y) \leqslant \varepsilon, \forall i \geqslant 0\} \end{cases}$$

for $x \in T_i$, some small $\varepsilon > 0$

are sub-manifolds of T_i (with $\mathrm{Dim}\, W^s(x,T_i) = \dim E^s$ and $\mathrm{Dim}\, W^u(x,T_i) = \dim E^u$) and satisfy

$$(*) \qquad \begin{cases} P(\mathrm{int}\, W^s(x,T_i)) \subset \mathrm{int}\, W^s(Px, T_j) & \text{for some } j = j(x) \\ P^{-1}(W^u(x,T_i)) \subset W^u(P^{-1}x, T_m) & \text{for some } m = m(x) \end{cases}$$

and r is constant on each $\mathrm{int}\, W^s(x,T_i)$.

(These constructions are due to Ratner, Bowen - after Sinai)

Step 2 : (The expanding map). — We define an equivalence relation on each T_i by $x \sim y$ if $W^s(x,T_i) = W^s(y,T_i)$ and denote $\Sigma = \bigcup_i T_i/\sim$. By hypothesis (*) the induced map $f : \Sigma \longrightarrow \Sigma$, $f([x]) = [Px]$ is well-defined.

THEOREM (Bowen). — *There is (almost) a one-one correspondence between f-periodic orbits $\{x, fx, \ldots, f^{n-1}x\}$ and closed ϕ-orbits τ (of least period $\lambda(\tau) = \sum_{i=0}^{n-1} r(f^i x) =: r^n(x)$). There is (almost) a one-one correspondence between ϕ-invariant probability measures and f-invariant probability measures.*

But the real reason for introducing this method is the following :

THEOREM (Ruelle). — *If we define an operator*

$$\begin{cases} L_t : C^\alpha(\Sigma) \longrightarrow C^\alpha(\Sigma) \\ (L_t g)(x) = \sum_{f(y)=x} g(y) \cdot e^{tr(y)} \quad t \in \mathbf{R} \end{cases}$$

then :

(i) L_t *has an (isolated) maximal simple positive eigenvalue* $e^{P(t)}$, *with* $t \mapsto e^{P(t)} \in (0, \infty)$ *strictly monotonely increasing and* $t_0 = h(\phi)$ iff $e^{P(t_0)} \equiv 1$ *is the maximal eigenvalue ;*

(ii) If

$$\begin{cases} L_{t_0} k = k \in C^\alpha(\Sigma) \\ L_{t_0}^* \overline{\nu} = \overline{\nu} \in C^\alpha(\Sigma)^* \end{cases}$$

are the positive eigenfunction and its dual then $\overline{\mu} = k\overline{\nu}$ *is a* f*-invariant (probability) measure corresponding to the maximal measure* μ *for* ϕ *(and the transverse measure constructed by Margulis) ;*

(iii) For each $z \in \Sigma, t \in \mathbf{R}$ *or* $t \in \mathbf{C}$ $\quad (L_t^n 1)(z) = \sum_{f^n(y)=z} e^{tr^n(y)} \sim \sum_{f^n(x)=x} e^{tr^n(x)}$
(and this can be used to study the zeta function of the flow ϕ*).*

Reference

[PP] PARRY W., POLLICOTT M. — *Zeta functions and the periodic structure of hyperbolic dynamics*, Astérisque, (1990), 187–188.