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ON THE DIRICHLET PROBLEM AT INFINITY FOR MANIFOLDS OF NON-POSITIVE CURVATURE (résumé)

by *Werner BALLMANN*

A complete Riemannian manifold M^n does not admit non-constant harmonic functions in L_p for $1 < p < \infty$, see [Y 2]. As for $p = \infty$, this also holds if the sectional curvature of M is non-negative [Y 1]. The case of non-positive curvature is discussed below. We assume that M is simply connected. Then M is homeomorphic to the unit disc and has a natural compactification by a sphere at infinity, $M(\infty)$.

Consider first the case of a symmetric space $M = G/K$ of non-compact type. Then the space of bounded harmonic functions on M was described by Fürstenberg [F]. It is naturally isomorphic to the space of bounded measurable functions on G/P, where $P \subset G$ is minimal parabolic. Note that $\dim(G/P) = n - \operatorname{rank}(M) \leqslant n - 1$.

Next consider the Dirichlet problem at infinity : given a continuous function f on $M(\infty)$, is there a harmonic function on M which extends continuously to f at $M(\infty)$. If M is a symmetric space of non-compact type, the Dirichlet problem at infinity (for arbitrary f) can be solved iff $\operatorname{rank}(M) = 1$. This is quite clear from the result of Fürstenberg mentioned above. More generally, if the curvature K of M is strictly negative, $-a^2 \leqslant K \leqslant -b^2 < 0$, then the Dirichlet problem at infinity is solvable, as was shown by Anderson [Ad] and Sullivan [S] (compare also [K 1]).

Suppose now that M admits a discrete group of isometries F such that M/F is compact. If M is not a Riemannian product, then M is a symmetric space of non-compact type and $\operatorname{rank}(M) > 1$, or M admits a geodesic which has no non-zero perpendicular parallel Jacobi field [B 2, BS]. In the latter case we say that M is a space of rank one. Such spaces were studied in [B 1], and in many respects they resemble spaces of negative curvature.

THEOREM [B 3]. — *Suppose M admits a discrete group of isometries F such that M/F is compact. If M has rank one, then the Dirichlet problem at infinity is solvable.*

In the proof it is shown that Brownian motion starting at a point $p \in M$ converges at $M(\infty)$ and that the hitting measure tends weakly to the Dirac measure at $z \in M(\infty)$ if p approaches z .

In the case of strictly negative curvature, more detailed results on harmonic functions were obtained by Anderson and Schoen [AS]. They showed, among others, that $M(\infty)$ is naturally isomorphic to the Martin boundary of M. Most of their results have been reproved (and partly extended) by other means by Ancona [Ac] and Kifer [K 2].

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