

# SÉMINAIRE DE THÉORIE SPECTRALE ET GÉOMÉTRIE

KAZUYOSHI KIYOHARA

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*Séminaire de Théorie spectrale et géométrie*, tome 6 (1987-1988), p. 33-37

[http://www.numdam.org/item?id=TSG\\_1987-1988\\_\\_6\\_\\_33\\_0](http://www.numdam.org/item?id=TSG_1987-1988__6__33_0)

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## ON BLASCHKE MANIFOLDS AND HARMONIC MANIFOLDS

by Kazuyoshi KIYOHARA

**0.** — A compact riemannian manifold  $M$  is called a Blaschke manifold if the diameter of  $M$  and the injectivity radius of  $M$  coincide. It is known that if  $M$  is a Blaschke manifold, then  $M$  is diffeomorphic to  $S^n$  or  $\mathbf{R}P^n$ , or  $\pi_1(M) = \{0\}$  and  $H^*(M, \mathbf{Z}) \cong$  the  $\mathbf{Z}$ -cohomology ring of  $\mathbf{C}P^n$ ,  $\mathbf{H}P^n$ ,  $\mathbf{C}aP^2$ .

The main problem about Blaschke manifolds is to know if the following conjecture, the Blaschke conjecture, is true or not : if  $M$  is a Blaschke manifold, then it would be a compact rank one symmetric space.

There are classes of riemannian manifolds related to Blaschke manifolds. A riemannian manifold  $M$  is called a globally harmonic manifold if the determinant of  $d(\exp_p)_x : T_p M \rightarrow T_{\exp_p x} M$  ( $p \in M$ ,  $x \in T_p M$ ) depends only on the norm  $|x|$ . A compact riemannian manifold is called a  $C_1$ -manifold if all of its geodesics are closed and have the same length 1. The relation is as follows :

$$\begin{array}{l} \text{compact, simply connected,} \\ \text{globally harmonic} \end{array} \implies \text{Blaschke} \implies C_1 .$$

The following results are known :

1. (Green, Berger et al.). — If  $(S^n, g)$  is a Blaschke manifold, then it is isometric to the standard one.

2. (Green, Berger et al.). — If  $(\mathbf{R}P^n, g)$  is a  $C_1$ -manifold, then it is isometric to the standard one.

3. (Kiyohara). — Let  $P$  be one of the projective spaces  $\mathbf{C}P^1$ ,  $\mathbf{H}P^n$  ( $n \geq 2$ ),  $\mathbf{C}aP^2$ , and let  $(P, g)$  be a  $C_\pi$ -manifold. If the metric  $g$  is sufficiently close to the standard  $C_\pi$ -metric  $g_0$ , then  $(P, g)$  is isometric to the standard one  $(P, g_0)$ .

4. (Zoll, Weinstein). — There are non-standard  $C_1$ -manifolds  $(S^n, g)$  for any dimension  $n \geq 2$ .

1. — From now on we assume  $M$  is a Blaschke manifold,  $\pi_1(M) = \{0\}$ ,  $H^*(M, \mathbf{Z}) \cong H^*(\mathbf{C}P^n, \mathbf{Z})$  ( $\dim M = 2n$ ,  $n \geq 2$ ), and the diameter of  $M$  is  $\pi/2$ . The followings are known about  $M$ :

1) For any  $p \in M$  and any  $q \in \text{Cut}(p)$  (the cut locus of  $p$ ), the distance  $d(p, q) = \pi/2$ .

2) Every cut locus is a submanifold of codimension 2.

3) Let  $\rho$  be the bundle projection  $TM \rightarrow M$ , and let  $\{\zeta_t\}$  be the geodesic flow on  $SM$ . Then  $\rho \circ \zeta_{\pi/2} : S_p(M) \rightarrow \text{Cut}(p)$  is a fibre bundle whose fibres are great circles on  $S_p M$ .

4) For  $p, q \in M$  with  $d(p, q) = \pi/2$ , we denote by  $\Sigma(p, q)$  the union of geodesic orbits through  $p$  and  $q$ . Then  $\Sigma(p, q)$  is a 2-dimensional submanifold diffeomorphic to  $S^2$ .

Now we define a mapping  $I : SM \rightarrow SM$  as follows: since  $H_2(M, \mathbf{Z}) \cong \mathbf{Z}$ , we fix a positive generator. Then on each  $\Sigma(p, q)$  the orientation is determined. Hence we have an orientation on each fibre  $S^1$  of the fibre bundle  $\rho \circ \zeta_{\pi/2} : S_p M \rightarrow \text{Cut}(p)$ , because the fibre  $S^1$  over  $q \in \text{Cut}(p)$  is nothing but the unit sphere of  $T_p \Sigma(p, q)$ . So  $I : SM \rightarrow SM$  is defined by the conditions:

- 1) If  $v \in S_p M$ , then  $Iv \in S_p M$  and  $\rho(\zeta_{\pi/2} v) = \rho(\zeta_{\pi/2} Iv)$ .
- 2)  $\langle v, Iv \rangle = 0$ .
- 3)  $\{v, Iv\}$  is positive in this order.

We extend the mapping  $I$  to  $TM \setminus \{0\}$ -section homogeneously, and let  $I_{*v} : T_{\rho(v)} M \rightarrow T_{\rho(v)} M$  be the differential of  $I|_{T_{\rho(v)} M \setminus \{0\}}$  at  $v$ . From the definition the mapping  $I$  satisfies  $I \circ I = (-1)$  identity. So it looks like an almost complex structure, and we have the following

**PROPOSITION A.** — Assume  $I_{*v}^2 + 1 = 0$  for all  $v \in SM$ . Then  $I : T_p M \setminus \{0\} \rightarrow T_p M \setminus \{0\}$  can be extended to a linear mapping on  $T_p M$  for every  $p \in M$ , i.e.  $I$  is an almost complex structure and it is integrable. Therefore  $(M, I)$  is a hermitian manifold. Moreover each cut locus is a complex submanifold and is holomorphically isomorphic to  $\mathbf{C}P^{n-1}$ .

**PROPOSITION B.** — Assume  $\dim M = 4$ . If  $I_{*v}^2 + 1 = 0$  for all  $v \in SM$  and if every cut locus is minimal, then  $M$  is isometric to  $(\mathbf{C}P^2, g_0)$ .

**LEMMA C.** — If  $M$  is moreover globally harmonic, then  $(I_{*v}^2 + 1)^{n-1} = 0$  for every  $v \in SM$  and every cut locus is minimal ( $\dim M = 2n$ ).

**COROLLARY D.** — If  $\dim M = 4$  and  $M$  is globally harmonic, then  $M$  is isometric to  $(\mathbf{C}P^2, g_0)$ .

*Remarque.* — This corollary is already known by a different method. See [1].

2. — For the proof of propositions we need some lemmas.

LEMMA 1. — *There is a Jacobi field  $Y(t)$  along the geodesic  $\gamma_v(t) = \rho(\zeta_t v)$  such that*

$$\begin{bmatrix} Y(0) \\ Y'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ Iv \end{bmatrix} \quad , \quad \begin{bmatrix} Y(\pi/2) \\ Y'(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 \\ -I\bar{v} \end{bmatrix} \quad , \quad \bar{v} = \zeta_{\pi/2} v .$$

*Moreover if a Jacobi field  $X(t)$  along  $\gamma_v(t)$  satisfies  $X(0) = X(\pi/2) = 0$  then  $X(t)$  is a constant multiple of  $Y(t)$ .*

For  $X, Y \in T_p M$ ,  $Y \neq 0$ , we put  $\nabla_X I \cdot Y = \nabla_{\partial/\partial t}(IY_t)|_{t=0}$ , where we take a curve  $c(t)$  in  $M$  such that  $c'(0) = X$ , and  $Y_t$  is the parallel displacement of  $Y$  along  $c(t)$ .  $\nabla_X I \cdot Y$  is linear in  $X$ , but not necessarily in  $Y$ .

LEMMA 2. — *Let  $Y(t)$  be a periodic Jacobi field along the geodesic  $\gamma_v(t)$ ,  $v \in SM$ . Then we have a periodic Jacobi field  $Z(t)$  along the geodesic  $\gamma_{e^{sI}v}(t)$  ( $e^{sI}v = v \cos s + Iv \sin s$ ) such that*

$$\begin{bmatrix} Z(0) \\ Z'(0) \end{bmatrix} = \begin{bmatrix} Y(0) \\ (\cos s + \sin s I_{*v})Y'(0) + \sin s(\nabla I \cdot v)Y(0) \end{bmatrix}$$

$$\begin{bmatrix} Z(\pi/2) \\ Z'(\pi/2) \end{bmatrix} = \begin{bmatrix} Y(\pi/2) \\ (\cos s + \sin s I_{*v})Y'(\pi/2) - \sin s(\nabla I \cdot \bar{v})Y(\pi/2) \end{bmatrix} ,$$

where  $(\nabla I \cdot v)Y(0) = \nabla_{Y(0)} I \cdot v$ , etc.

LEMMA 3. —

1) *There are Jacobi fields  $Y_1(t)$ ,  $Y_2(t)$  along  $\gamma_v(t)$  such that*

$$\begin{bmatrix} Y_1(0) \\ Y_1'(0) \end{bmatrix} = \begin{bmatrix} Iv \\ -\nabla_v I \cdot v \end{bmatrix} \quad , \quad \begin{bmatrix} Y_1(\pi/2) \\ Y_1'(\pi/2) \end{bmatrix} = \begin{bmatrix} -I\bar{v} \\ \nabla_{\bar{v}} I\bar{v} \end{bmatrix}$$

$$\begin{bmatrix} Y_2(0) \\ Y_2'(0) \end{bmatrix} = \begin{bmatrix} 2\nabla_v I \cdot v \\ R(Iv, v)v - \nabla_v^2 I \cdot v \end{bmatrix} \quad , \quad \begin{bmatrix} Y_2(\pi/2) \\ Y_2'(\pi/2) \end{bmatrix} = \begin{bmatrix} -2\nabla_{\bar{v}} I \cdot \bar{v} \\ -R(I\bar{v}, \bar{v})\bar{v} + \nabla_{\bar{v}}^2 I \cdot \bar{v} \end{bmatrix} .$$

2)  $\nabla_{e^{sI}v} I \cdot d^{sI}v = \nabla_v I \cdot v$ .

LEMMA 4. — *Let  $Y(t)$  be a periodic Jacobi field along  $\gamma_v(t)$ . Then there is a periodic Jacobi field  $Z(t)$  along  $\gamma_v(t)$  such that*

$$\begin{bmatrix} Z(0) \\ Z'(0) \end{bmatrix} = \begin{bmatrix} {}^t I_{*v} Y(0) \\ I_{*v} Y'(0) + (\nabla I \cdot v - {}^t \nabla I \cdot v)Y(0) + \{(Y(0), \nabla_v I \cdot v) + (Y'(0), Iv)\}v \end{bmatrix}$$

$$\begin{bmatrix} Z(\pi/2) \\ Z'(\pi/2) \end{bmatrix} = \begin{bmatrix} {}^t I_{*\bar{v}} Y(\pi/2) \\ -I_{*\bar{v}} Y'(\pi/2) - (\nabla I\bar{v} - {}^t \nabla I\bar{v})Y(\pi/2) - \{(Y(\pi/2), \nabla_{\bar{v}} I\bar{v}) + (Y'(\pi/2), I\bar{v})\}\bar{v} \end{bmatrix} .$$

**3. Proof of Proposition A.** — Fix  $p \in M$  and consider the  $S^1$ -principal bundle  $\rho \circ \zeta_{\pi/2} : S_p M \rightarrow \text{Cut}(p)$ , where the  $S^1$ -action is given by  $e^{sI}$ ,  $0 \leq s \leq 2\pi$ .

We define a 1-form  $\omega$  on  $S_p M$  by

$$\omega(X) = \langle X, Iv \rangle, \quad X \in T_v(S_p M) = \{Y \in T_p M \mid \langle v, Y \rangle = 0\}.$$

As is easily seen,  $\omega$  is a connection form, *i.e.* invariant under the  $S^1$ -action. We have

$$d\omega(X, Y) = \langle (I_{*v} - {}^t I_{*v})X, Y \rangle.$$

So there is a unique closed 2-form  $\Omega$  on  $\text{Cut}(p)$  such that  $(\rho \circ \zeta_{\pi/2})^* \Omega = d\omega$ . We can see that  $[(1/2\pi)\Omega]$  is a generator of  $H^2(\text{Cut}(p), \mathbf{Z}) \cong \mathbf{Z}$ . Therefore

$$(1/2\pi)^{n-1} \int_{\text{Cut}(p)} \Omega^{n-1} = 1$$

under a proper orientation of  $\text{Cut}(p)$ , and thus

$$\int_{S_p M} \omega \wedge (d\omega)^{n-1} = (2\pi)^n.$$

Now put  $J_v = I_{*v} - {}^t I_{*v}$ ,  $S_v = I_{*v} + {}^t I_{*v}$ . Then  $2I_{*v} = J_v + S_v$  and

$$I_{*v}^2 + 1 = 0 \iff J_v^2 + S_v^2 + 4 + J_v S_v + S_v J_v = 0. \quad (\#)$$

Let  $e_1, \dots, e_{2n-2}$  be an orthonormal basis of the orthogonal complement to  $\mathbf{R}v + \mathbf{R}Iv$  in  $T_p M$  such that  $J_v e_{2i-1} = \lambda_i e_{2i}$ ,  $J_v e_{2i} = -\lambda_i e_{2i-1}$ ,  $\lambda_i \geq 0$ ,  $i = 1, \dots, n-1$ . By (#) we have

$$-\lambda_i^2 + |S_v e_{2i}|^2 + 4 = 0.$$

Hence  $\lambda_i \geq 2$ , and  $\lambda_i = 2$  for every  $i$  if and only if  $S_v = 0$ . Then

$$(\omega \wedge (d\omega)^{n-1})(Iv, e_1, \dots, e_{2n-2}) = (n-1)! \prod_{i=1}^{n-1} \lambda_i \geq 2^{n-1} (n-1)!,$$

and the equality holds if and only if  $S_v = I_{*v} + {}^t I_{*v} = 0$ . Therefore we have

$$(2\pi)^n = \int_{S_p M} \omega \wedge (d\omega)^{n-1} \geq 2^{n-1} (n-1)! \text{vol}(S_p M).$$

But  $\text{vol}(S_p M)$  is just  $2\pi^n / (n-1)!$ . So the equality holds in the above inequality. Hence we have  $S_v = I_{*v} + {}^t I_{*v} = 0$  for any  $v \in SM$ . Since  $I_{*v}^2 + 1 = 0$ , it follows that  ${}^t I_{*v} I_{*v} = 1$ . This implies that the mapping  $I : S_p M \rightarrow S_p M$  is an isometry, and therefore the restriction of a linear orthogonal transformation of  $T_p M$ . Hence  $I$  is extended as a tensor field of type (1,1) with  $I^2 = -1$ , *i.e.* an almost complex structure on  $M$ , and  $(M, I)$  is an almost hermitian manifold.

By using the square of the endomorphisms on the space of Jacobi fields in Lemma 4, one gets

$$\langle \{I(\nabla I \cdot v - {}^t \nabla I \cdot v) - (\nabla I \cdot v - {}^t \nabla I \cdot v)I\} X, Y \rangle = 0, \quad X, Y \perp v, Iv.$$

Moreover Lemma 3 (2) gives

$$\langle \{I(\nabla I \cdot v + {}^t \nabla I \cdot v) - (\nabla I \cdot v + {}^t \nabla I \cdot v)I\} X, Y \rangle = 0, \quad X, Y \perp v, Iv.$$

These formula gives

$$\nabla_{IX}I = I\nabla_X I$$

for any vector  $X$ . By this it is easy to see that the Nijenhuis' tensor vanishes, and  $(M, I)$  turns out to be a hermitian manifold.

It is now clear that each cut locus is a complex submanifold of  $M$  and the  $S^1$ -fibration  $\rho \circ \zeta_{\pi/2} : S_p M \rightarrow \text{Cut}(p)$  is nothing but the standard Hopf fibration :  $S^{2n-1} \rightarrow CP^{n-1}$ . Hence the last statement of the proposition follows.

*Proof of Proposition B.* — For  $v \in SM$  we define the symmetric endomorphism  $\Phi_v$  of  $T_{\rho(v)}M$  by  $\Phi_v v = \Phi_v I v = 0$  and

$$\langle \Phi_v X, Y \rangle = -\langle h(X, Y), v \rangle, \quad X, Y \in T_{\rho(v)}M, \quad X, Y \perp v, I v,$$

where  $h$  is the second fundamental form of  $\text{Cut}(\rho\zeta_{\pi/2}v)$  in  $M$  at  $\rho(v)$ . If we take a curve  $c(t)$  in  $\text{Cut}(\rho\zeta_{\pi/2}v)$  such that  $c'(0) = X$ , and a normal vector field  $v_t$  to  $\text{Cut}(\rho\zeta_{\pi/2}v)$  along  $c(t)$ , we have

$$\langle \Phi_v X, Y \rangle = \langle \nabla_{\partial/\partial t} v_t |_{t=0}, Y \rangle.$$

So the following lemma is clear.

LEMMA 5. —  $\Phi_{Iv}X = I\Phi_v X + (\nabla_X I)v$ ,  $X \in T_{\rho(v)}M$ ,  $X \perp v, Iv$ .

Since every cut locus is minimal, it follows that  $\text{tr } \Phi_v = 0$  for any  $v \in Sm$ ,  $\text{tr}$  being the trace. Hence in view of Lemma 5 one gets

$$\text{tr } (\nabla I)v = 0.$$

This together with the formula  $\nabla_{IX}I = I\nabla_X I$ , shown in the proof of Proposition A, implies that  $\nabla I = 0$ , i.e.  $(M, I)$  is kählerian.

By applying Lemma 1 to the Jacobi field  $Y_2$  in Lemma 3,

$$R(Iv, v)v = c(v)Iv, \quad v \in SM,$$

where  $c$  is a function on  $SM$  satisfying  $c(\zeta_{\pi/2}v) = c(v)$ . As is easily seen,  $c(v)$  is pointwise constant, i.e. if  $v_1$  and  $v_2$  are based at the same point on  $M$ , then  $c(v_1) = c(v_2)$ . Using the fact that for any two points  $p$  and  $q$  on  $M$ , there is a point  $m$  such that  $d(p, m) = d(q, m) = \pi/2$ , we see the constancy of  $c(v)$ .

Since  $(M, I)$  is kählerian and has constant holomorphic sectional curvature, it must be holomorphically isometric to  $(CP^2, g_0)$ .

Lemma C is an immediate consequence of Lemma 4.

### Reference

- [1] BESSE A. — *Manifolds all of whose geodesics are closed*, Springer, 1978.

Kazuyoshi KIYOHARA  
 Max-Planck-Institut für Mathematik  
 Gottfried-Claren-Strasse 26  
 D - 5300 BONN 3  
 (R.F.A.)