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THE SEMI-CLASSICAL ERGODIC THEOREM FOR DISCONTINUOUS METRICS

Yves Colin de Verdière

ABSTRACT. — In this paper, we present an extension of the classical Quantum ergodicity Theorem, due to Shnirelman, to the case of Laplacians with discontinous metrics along interfaces. The "geodesic flow" is then no more a flow, but a Markov process due to the fact that rays can by reflected or refracted at the interfaces. We give also an example build by gluing together two flat Euclidean disks.

Introduction

Recently, Dmitry Jakobson, Yuri Safarov and Alexander Strohmaier, in the paper [9], proved a *Quantum Ergodicity Theorem* (denoted QE in what follows) for piecewise smooth Riemannian manifolds. In this lecture, I will present the result as well as the main ideas of the proof. I will also provide a simple example where the result applies. The message here is that, due to the possibilities of reflexion or refraction of waves along the hypersurface of discontinuity of the metric, the geodesic flow is no more deterministic, but can be viewed as *random process*. Ergodicity makes sense for such a Markov process. The main result can be summarized as follows: *if the geodesic flow is ergodic and if there are few recombining geodesics, then we have QE.*

Let us first recall the standard *Quantum Ergodicity* Theorem, due essentially to A. Shnirelman (see [11, 16, 13]):

THEOREM. — *Let (X, g) be a smooth closed Riemannian manifold and assume that the geodesic flow of (X, g) is ergodic. Let us denote by $(\phi_j)_{j=1,\cdots}$ an orthonormal eigen-basis of $L^2(X, |dx|_g)$ with $\Delta_g \phi_j = \lambda_j \phi_j$. Then there exists a density one sub-sequence $(\lambda_{j_k})_{k=1,\cdots}$ of the sequence of eigenvalues so that, for any pseudo-differential operator A of degree 0 on*

X, we have

$$\lim_{k\to\infty}\langle A\phi_{j_k}|\phi_{j_k}\rangle = \int_{S^\star X}\sigma(A)dL$$

where $\sigma(A): T^\star X \to \mathbb{R}$ is the principal symbol of A and dL is the normalized Liouville measure on the unit cotangent bundle $S^\star X$.

The previous result applies in particular to manifolds of <0 sectional curvature. A more intuitive corollary is

Corollary. — *If $D \subset X$ is a domain with piecewise smooth boundary,*

$$\lim_{k\to\infty}\int_D |\phi_{j_k}|^2|dx|_g = \frac{|D|}{|X|}$$

where $|D|$ denotes the $g-$volume of D.

This result has been extended to manifolds with boundary in [8, 17] and also to sub-Riemannian Laplacians of contact type in dimension 3 in [15]. The extension to piecewise smooth metrics proposed in [9] is more subtle, because on such manifolds the geodesic flow is not a classical flow associated to a vector field: *a ray arriving on an interface splits into a reflected and a refracted ray.* To such a situation there is a naturally attached Markov process, also called the geodesic flow, describing the propagation of the energy of high frequency waves. The ergodicity of the geodesic flow as a Markov process is a natural assumption for an extension of the QE Theorem. However, a piece is missing, because the propagation of the energy is well defined only if there is no interferences between different geodesics which coïncide outside a finite time interval. So that another assumption is needed: such pairs of "recombining geodesics" are of measure 0. Hopefully this assumption is generically satisfied. However, as we will see, it is not satisfied in the very close context of Quantum Graphs. In this case, it is known that QE does not hold: see [4] for star graphs and [14] for the general case.

1. Laplace-Beltrami operators for discontinuous metrics

Definition 1.1. — *If X is a smooth closed manifold of dimension d, a measurable Riemannian metric g on X is* uniform *if, for a smooth Riemannian metric g_0 on X, there exist two constants C_1 and C_2, with $0 < C_1 < C_2$, so that $0 < C_1 g_0 \leqslant g \leqslant C_2 g_0$.*

The Laplace operator Δ_g on (X, g) is the self-adjoint operator on $L^2(X, |dx|_g)$ defined as the Friedrichs extension *of the closed quadratic* *form* $Q(f) = \int_X \|df\|_g^2 |dx|_g$ *whose domain is the Sobolev space* $H^1(X)$*, which is independent of* g *as soon as* g *is uniform.*

QUESTION 1.2. — *Is the Weyl asymptotic formula valid for such a metric?*

DEFINITION 1.3. — *Let us give a smooth closed manifold* X*. A* piecewise smooth Riemannian metric g *on* X *is a (uniform) Riemannian metric which is smooth outside a closed smooth hyper-surface* $X_{\rm sing}$ *of* X *and so that the metric* g *extends smoothly from both sides of the open set* $X_{\rm reg} := X \setminus X_{\rm sing}$ *to the metric completion of* $X_{\rm reg}$*. The metric* g *is in general discontinuous on* $X_{\rm sing}$*.*

The previous definition can be extended to cases where X is a simplicial complex which is not a manifold, for example X can be a graph viewed as a 1D singular manifold, such a graph is sometimes called a *Quantum graph* or a *metric graph*.

If (X, g) is a piecewise smooth Riemannian metric, the Laplace-Beltrami operator Δ_g can be defined as in the Definition 1.1.

PROPOSITION 1.4. — *The domain of* Δ_g *is the space of functions* f *on* X *which are in the Sobolev spaces* $H^1(X)$ *and* $H^2(X_{\rm reg})$ *and whose weighted sum of the two normal derivatives at any point of the smooth part of* $X_{\rm sing}$ *vanishes where the weights at the point* x *of* $X_{\rm sing}$ *are the densities of the Riemannian volume of the limit metrics on the corresponding sides.*

We call these conditions on the behavior of f across $X_{\rm sing}$ the *continuity conditions.*

2. Propagation of waves across $X_{\rm sing}$

In order to see the effect of the continuity conditions on the wave propagation, we take a simple example with constant coefficients. This will give the rules for the propagation of the high frequency waves along the geodesics.

Let us consider on $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_y$ the metric g given by $g = n_+^2(dx^2 + dy^2)$ on $y > 0$ and $g = n_-^2(dx^2 + dy^2)$ on $y < 0$. An incoming plane wave with speed 1 in $y > 0$ is defined by $u_{\rm in}(x, y) = \exp(i(x\xi - y\eta))$ with $\eta > 0$ and with $\xi^2 + \eta^2 = n_+^2$.

In order to satisfy the continuity condition we have to add to u_+ a reflected wave u_r on $y > 0$ and a refracted wave u_ρ on $y < 0$ of the

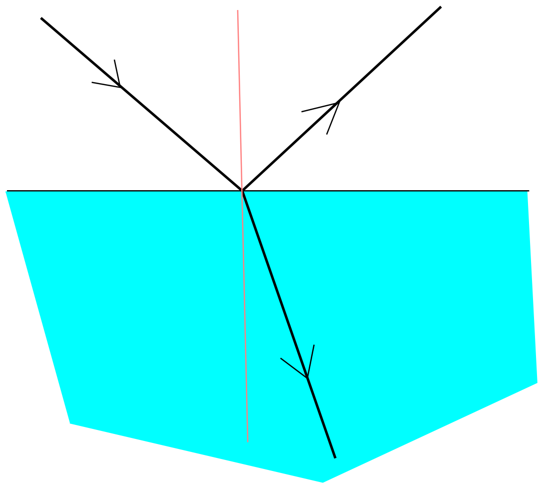

Figure 2.1. an incident geodesic with the reflected and the refracted rays.

following forms:

$$u_r = t_r e^{i(x\xi+y\eta)},\ u_\rho = t_\rho e^{i(x\xi-y\eta')}$$

with $\eta' > 0$ and $\xi^2 + \eta'^2 = n_-^2$. We have total reflexion if $|\xi| > n_-$. In this case the refracted wave is exponentially decaying. The angles $\iota_\pm$ of the rays with the normals to the $y-$axis satisfy $\sin\iota_\pm = \xi/n_\pm$ which gives the Snell law

$$n_+ \sin\iota_+ = n_- \sin\iota_- \ .$$

The coefficients t_r and t_ρ have to satisfy: $1 + t_r = t_\rho$ (continuity of u along $y = 0$) and $n_+\eta(1 - t_r) = n_- t_\rho \eta'$ (vanishing of the sum of the weighted normal derivatives). This allows to compute t_r and t_ρ:

$$t_r = \frac{1-\alpha}{1+\alpha},\ t_\rho = \frac{2}{1+\alpha} \text{ with } \alpha = \frac{n_-\eta'}{n_+\eta_-} \ .$$

The conservation of the energy density can be checked as follows: take a compact domain $D \subset Y_+$ and let D_r and D_ρ be the domains obtained from D following the rays associated to the incident plane wave during a large enough time. Then

$$\int_D |du_{\text{in}}|^2 |dx_{g^+}| = \int_{D_r} |du_r|^2 |dx_{g^+}| + \int_{D_\rho} |du_\rho|^2 |dx_{g^-}| \ .$$

Summarizing, we have:

- The refracted wave is exponentially decaying if $|\xi| > n_-$.

- The angles of the incident and refracted rays with the normals satisfy the Snell law.
- The conservation of energy expresses as

$$t_r^2 + \alpha t_\rho^2 = 1 \ .$$

We define $p_r = t_r^2$ and $p_\rho = \alpha t_\rho^2 = 1 - p_r$.

The propagation of waves is given formally by the "unitary wave group" $U(t) := \exp(-it\sqrt{\Delta})$. The propagation of high frequency waves along geodesics arriving transversely to $X_{\rm sing}$ is described as a sum of two Fourier Integral Operator's: $U(t) = U_r(t) + U_\rho(t)$. The associated canonical transformations $\Phi_r(t)$ and $\Phi_\rho(t)$ are the reflected and the refracted geodesic flows. The symbols satisfy the usual transport equations in $X_{\rm reg}$ and are multiplied by t_r (resp. t_ρ) for the reflected (resp. refracted wave). The proof follows Chazarain's method [5]. We can apply Egorov Theorem with A being a pseudo-differential operator of principal symbol a: the operators $U_r(-t)AU_r(t)$ (resp. $U_\rho(-t)AU_\rho(t)$) are pseudo-differential operators of principal symbols $p_r a\,(\Phi_r(t))$ (resp. $p_\rho a\,(\Phi_\rho(t))$).

3. The geodesic flow as a Markov process

We will make the following Assumptions:

(1) *The set of geodesics which are not defined for all times has measure* 0.

(2) *The set of geodesics which are tangent for some time to the hypersurface of discontinuities of the metric has measure* 0.

The Assumptions (2) is always true. The Assumption (1) is probably true in the generic case; the bad geodesics hit the $X_{\rm sing}$ infinitely many times in a finite time interval!

If a geodesic starting in $X_{\rm reg}$ hits $X_{\rm sing}$ transversely, it can be reflected or refracted, or totally reflected. We associate probabilities to both events as follows:

Definition 3.1. — *The probability of being reflected is given by $p_r = |t_r|^2$, with t_r defined in Section 2, and the probability of being refracted is given by $p_\rho = 1 - p_r$.*

Remark 3.2. — Note that p_r and p_ρ are functions on the unit ball bundles $Y_\pm$ of $T^\star X_{\rm sing}$. Moreover, p_r (resp. p_ρ) can be extended by 0 on $Y_- \setminus Y_+$ (resp. on $Y_+ \setminus Y_-$) and these extensions are continuous.

This way the *geodesic flow* is a well defined Markov process on the unit cotangent bundle denoted by Z. We will associate to the geodesic flow a semi-group of positive operators on $L^\infty(Z)$ defined as follows: to any geodesic $\gamma : [0,t] \to X$ crossing $X_{\rm sing}$ at a finite number of points, we associate a positive weight $w(\gamma)$ which is the product of the probability transitions at the crossing points. If $t > 0$ is fixed, almost all geodesics cross $X_{\rm sing}$ at a finite number of points on the interval $[0,t]$, and we define

$$G_t f(z) = \sum_{\gamma\in\Omega,\ \gamma(0)=z} w\left(\gamma_{|[0,t]}\right) f(\gamma(t)) \ .$$

We have $G_{t+s} = G_t \circ G_s$ and $G_t 1 = 1$.

Definition 3.3. — *If ω is the symplectic form on $T^\star X$, the* Liouville measure *on the unit cotangent bundle is the measure $\left|\wedge^d \omega / dg^\star\right|$, normalized so that it is a probability measure denoted dL.*

Proposition 3.4. — *The Liouville measure on the unit cotangent bundle is invariant by the geodesic flow: it means that $\int_Z G_t f dL = \int f dL$. In particular, G_t extends to a positive operator on $L^1(Z, dL)$ of norm 1.*

Definition 3.5. — *The geodesic flow is* ergodic *if and only if the only measurable functions which are invariant by the semi-group $(G_t)_{t\geqslant 0}$ are the functions which are constant outside a measure 0 set.*

As a Corollary of ergodicity, we get the

Proposition 3.6. — *If the geodesic flow is ergodic and $f \in L^1(Z, dL)$, we have, for almost all $z \in Z$,*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T G_t f(z) dt = \int_{S^\star X} f dL \ .$$

This is proved using the point-wise ergodic Theorem given in [7], Theorem 5, page 690 or in [10], Theorem 3.7, page 217:

Theorem 3.7. — *If $(G_t)_{t\geqslant 0}$ is a strongly measurable semi-group on $L^1(\mu)$ whose norms on $L^1(\mu)$ and $L^\infty(\mu)$ are bounded by 1, then, for $f \in L^1$, the averages*

$$\frac{1}{T}\int_0^T G_t f(z) dt$$

converge for almost all z as $T \to +\infty$. The limit function $z \to \bar{f}(z)$ is invariant by G_t for all t.

This last property will be crucial for the QE Theorem.

4. How to prove ergodicity?

4.1. Invariant sets

DEFINITION 4.1. — *A measurable subset A of Z is said to be* invariant *by the geodesic flow if, for almost all $z \in Z$ and for all $t > 0$, $G_t\chi_A(z) = \chi_A(z)$.*

LEMMA 4.2. — *Let $f : Z \to \mathbb{R}$ be an L^∞ function which is invariant by $G_t,\ t > 0$. Then f is measurable with respect to the $\sigma-$algebra generated by invariant sets.*

This is an easy consequence of Lemma 3.3, page 126 in [10]. From this, we get:

THEOREM 4.3. — *The geodesic flow is ergodic if and only if any set invariant by the geodesic flow is of Liouville measure* 0 *or* 1.

Using invariance for small values of t, one gets:

THEOREM 4.4. — *If A is an invariant set by the geodesic flow, then the set $\bar{A}$ of points which are on some smooth geodesic arc in X_{reg} crossing A satisfies $|\bar{A} \setminus A| = 0$.*

4.2. Poincaré maps

Let us assume for simplicity that X_{sing} cuts X into two open disjoint parts $X \setminus X_{\mathrm{sing}} = X_+ \cup X_-$. Let us denote by $Y_\pm$ the unit ball bundles for $g^\star_\pm$ in $T^\star X_{\mathrm{sing}}$. Let $z = (x, \eta) \in Y_+$. There is a unique geodesic γ in X_+ so that $\gamma(0) = (x, \xi)$ with $g^\star_+(\xi) = 1$ and $\xi_{|T_x X_{\mathrm{sing}}} = \eta$. Let $t > 0$ be the first return time of γ on X_{sing} and $(x', \xi') = \gamma(t)$. Then we define $P_+(z) = (x', \xi'_{|T_{x'} X_{\mathrm{sing}}})$. We define in a similar way P_-. It follows from the Poincaré recurrence Theorem that the map P_+ (resp. P_-) is defined for almost every point of Y_+ (resp. Y_-). Let us consider the traces $A_\pm$ of a set A on $Y_\pm$. Then, if A is invariant by the geodesic flow, $A_+ \cup A_-$ is equivalent to a set invariant by P_+ and P_- modulo sets of measure 0. We get:

THEOREM 4.5. — *If X_{sing} is non empty, the geodesic flow is ergodic if and only if almost all geodesics cross X_{sing} AND if any set $C \subset Y_+ \cup Y_-$ invariant by P_+ and P_- is of measure* 0 *or has a complement of measure* 0.

5. The main result: semi-classical ergodicity for ray-splitting billiards

Let us start with the

Definition 5.1. — *Two geodesics $\gamma : \mathbb{R} \to X$ are called* recombining geodesics *if they coïncide outside a compact interval of* $\mathbb{R}$.

We will use the genericity Assumptions described in Section 3 and the following one

(3) The set of Cauchy data of recombining classical trajectories has measure 0.

Remark 5.2. — Assumption (3) is probably generically true. It is however not true for graphs not homeomorphic to a circle or an interval. See Section 8 for an example where genericity is proved.

The Assumptions (1) to (3) are probably generically true. Assumptions (1) and (2) are already present in the QE Theorem for manifolds with boundary. Assumption (3) is the more important: it is needed in order to be able to follow the propagation of the energy of waves by transport along the geodesic flow like in the Egorov Theorem.

Theorem 5.3. — *Under the Assumptions (1) to (3) and assuming that the geodesic flow is ergodic, there exists a sub-sequence S of density 1 of the set of eigenvalues so that for any pseudo-differential operator A of degree 0, compactly supported away from the hyper-surface of discontinuities of g and of principal symbol $\sigma(A)$, we have*

$$\lim_{j\to\infty,\ \lambda_j\in S} \langle A\phi_j|\phi_j\rangle = \int_{S^\star X} \sigma(A)dL \ .$$

Remark 5.4. — Assumption (3) is never satified for Quantum graphs not homeomorphic to the circle or the interval: there is no contradiction between the previous Theorem and the non validity of QE for almost all quantum graphs proved in [14].

6. The g-trace

The proof uses a regularized trace associated to the metric g, which we call the g-trace.

Let (X, g) be a closed Riemannian manifold with g a uniform metric. Let us denote by $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots$ the eigenvalues of the Laplace operator Δ_g with an associated orthonormal basis of $L^2(X,|dx|_g)$ of eigenfunctions $(\phi_j)_{j=1,\dots}$.

Definition 6.1. — *Let $A : L^2(X,|dx|_g) \to L^2(X,|dx|_g)$ be a bounded operator. We say that A admits a g-trace if the limit*

$$\mathrm{Tr}_g(A) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \langle A\phi_j|\phi_j\rangle \ .$$

The limit $\mathrm{Tr}_g(A)$ is called the g-trace of A.

As noticed in [9], the $g-$trace of a compact operator vanishes. This implies that if A is a Fourier integral operator the $g-$trace of A depends only of the principal symbol of A. The precise formula is given in the

Theorem 6.2. — *If g is a smooth Riemannian metric on X and if A is a Fourier integral operator of degree 0 on X associated to an homogeneous canonical diffeomorphism $\chi : T^\star X \setminus 0 \to T^\star X \setminus 0$, then the g-trace of A exists and is given by*

$$\mathrm{Tr}_g(A) = \int_{\mathrm{Fix}(\chi)\cap S^\star X} \sigma(A) dL \ ,$$

where $\mathrm{Fix}(\chi)$ is the set of fixed points of χ, dL is the normalized Liouville measure on the unit cotangent bundle $S^\star X$ of X and $\sigma(A) : T^\star X \setminus 0 \to \mathbb{C}$ is the principal symbol of A defined in the proof.

In particular, this formula applies if A is a pseudo-differential operator; then $\mathrm{Fix}(\chi) = T^\star X \setminus 0$ and $\sigma(A)$ is the usual principal symbol of A.

The authors of [9] called this result a *local Weyl formula*, because, I guess, by applying it to the operators of multiplication by a smooth function one gets that, if D is a smooth compact domain of X,

$$\lim_{\lambda\to+\infty} \frac{\sum_{\lambda_j \leqslant \lambda} \int_D |\phi_j|^2 |dx|_g}{\#\{j \mid \lambda_j \leqslant \lambda\}} = \frac{|D|}{|X|}.$$

Corollary 6.3. — *The same result applies to a singular metric g if A vanishes near the singular locus of g.*

Proof of Theorem 6.2. — The scheme of the proof is as follows: In part A, we prove the Theorem 6.2 with a absolutely continuous measure $d\tilde{L}$; in part B, we prove the Theorem 6.2 for pseudo-differential operators with the identification $d\tilde{L} = dL$; in part C, we prove, that the formula for pseudo-differential operators gives also the general formula.

Part A of the proof: For $|t|$ small enough, the Schwartz kernel of $U(t)A$ is given by

$$[U(t)A](x,y) = \int_{\mathbb{R}^d} e^{i\phi(t,x,y,\theta)} a(t,x,y,\theta)|d\theta|$$

where the phase function ϕ is generating function of $\Phi_t \circ \chi$ in the Hörmander sense where Φ_t is the geodesic flow, a is a smooth symbol of degree 0. We want to evaluate, similarly to what is done in [6], the sum

$$\Sigma(\mu) = \sum_{j=1}^{\infty} \rho(\mu - \mu_j)\langle A\phi_j|\phi_j\rangle$$

with $\mu_j = \sqrt{\lambda_j}$ and ρ is a positive Schwartz function whose Fourier transform

$$\hat{\rho}(t) = \int e^{-it\mu}\rho(\mu)d\mu$$

is positive, compactly supported near 0 and $\hat{\rho}(0) = 1$. We can rewrite

$$\Sigma(\mu) = \frac{1}{2\pi}\int e^{i(t\mu + \phi(t,x,x,\theta))}\hat{\rho}(t)a(t,x,x,\theta)|d\theta dx dt| \ .$$

We now make the change of variable $\theta = \mu r\omega$ with $r > 0$ and $\|\omega\| = 1$ and get

$$\Sigma(\mu) = \frac{\mu^d}{2\pi}\int e^{i(\mu(t + r\phi(t,x,x,\omega))}\hat{\rho}(t)a(t,x,x,\theta)r^{d-1}|dr dt d\omega dx| \ .$$

Let us show that we can apply the non degenerate stationary phase expansion to the integral w.r. to (r,t): the critical points are given by $1 + r\phi_t(t,x,x,\omega) = 0$, $\phi(t,x,x,\omega) = 0$ and the determinant of the corresponding Hessian is $-\phi_t^2$. The phase function ϕ satisfies the eiconal equation $\phi_t + H(x,\phi_x) = 0$ with $H = \sqrt{g^\star}$. Hence the Hessian is non degenerate. This way we get

$$\Sigma(\mu) = \mu^{d-1}\int_C e^{i(\mu(t + r\phi(t,x,x,\omega))}\hat{\rho}(t)\frac{a(t,x,x,\omega)}{|\phi_t(t,x,x,\omega)|}r^{d-1}|d\omega dx| \ , \tag{6.1}$$

where the integral is on the critical manifold C in (t,r). Let us look at the critical points in the new integral: their set is the set of (t,x,x,ω) in C_ϕ so that the corresponding point is a fixed point of χ. We get

(1) $\Sigma(\mu) = o\left(\mu^{d-1}\right)$ if the set of fixed points of χ is of measure 0,

(2)

$$\Sigma(\mu) = \mu^{d-1}\int_{\mathrm{Fix}(\chi)\cap\{H(x,\xi)=1\}} \sigma(A)(x,\xi)d\tilde{L} + o\left(\mu^{d-1}\right)$$

with $d\tilde{L}$ a suitable smooth measure to be determined now together with the meaning of the principal symbol $\sigma(A)$ of A.

Now a classical Tauberian Theorem, given in Appendix A, allows to conclude part A.

Part B of the proof: If $f : \mathbb{R}^d \setminus 0 \to \mathbb{R}$ is a smooth function which is homogeneous of degree $-d$, the differential form $\omega = f(\xi)\left(\sum_{j=1}^d \xi_j \widehat{d\xi_j}\right)$ is closed on $\mathbb{R}^d \setminus 0$ by Euler's formula. We use this with $f(\xi) = a(\xi)/H(x,\xi)^d$ and get by the Stokes formula:

$$\int_{S^{d-1}} \frac{a(\xi)}{H(x,\xi)^d} d\sigma(\xi) = \int_{H(x,\xi)=1} a(\xi)\alpha$$

with $\alpha = H^{-d}\left(\sum \xi_j \widehat{d\xi_j}\right)$. On $H=1$, we have also $\alpha = d\xi_1 \wedge \cdots d\xi_d / d_\xi H(x,\xi)$ by Euler formula.

If A is a pseudo-differential operator, the operator $U(t)A$ is a Fourier integral operator associated to the geodesic flow Φ_t. Following Hörmander, we can take $\omega(x,y,\xi) - tH(x,\xi)$ as a generating function where ω is a suitable generating function for the Identity map. We get

$$[U(t)A](x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\omega(x,y,\xi)-tH(x,\xi))} A(t,x,y,\xi) a(y,\xi) |d\xi|$$

(modulo compact operators), where $A(0,x,x,\xi) \equiv 1$ and a is the principal symbol of A. The critical set C of Part A is given by $1 - rH(x,\xi) = 1;\ t = 0$. From this, we get, by Equation (6.1),

$$\Sigma(\mu) = \frac{\mu^{d-1}}{(2\pi)^{d-1}} \int_{\mathbb{R}^d_x \times S^{d-1}_\omega} \frac{1}{H(x,\xi)^d} a(x,\omega) |dx d\omega| + o\left(\mu^{d-1}\right) .$$

In order to get the $g-$trace of A, we apply the Tauberian Theorem given in Appendix A.

Part C of the proof: in the fixed point set, there is a full measure set where the canonical transformation is tangent to the identity: this allows to reduce to the case of pseudo-differential operators. □

7. Sketch of the proof of Theorem 5.3

The main idea is that the unitary group $U(t) = \exp(it\sqrt{\Delta})$ (the wave flow) is a sum of FIO's. This is not exactly true due to the singularities of the metric and we omit the technical part of the work which consists in showing that these singularities problems are removable. Let us assume now that $U(t) = V_1(t) + V_2(t)$ where the V_j's are Fourier integral operators associated to classical flows $\psi_{j,t}$.

Let us take a pseudo-differential operator A of degree 0 with real principal symbol a so that the integral $\int_{S^\star X} adL$ vanishes. The operators $A_j(t) = V_j(-t)AV_j(t)$ are pseudo-differential operators and admits principal symbols $a_j(t)(z) = w_{t,j}(z)a(\psi_{j,t}(z))$ with $0 \leqslant w_{t,j} \leqslant 1$ and $w_{t,1} + w_{t,2} \leqslant 1$. The simplified classical ergodicity assumptions are now:

(1) For all t's, the measures of the set of fixed points of $\psi_{1,t} \circ \psi_{2,-t}$ vanish.

(2) If we define the operator W_t on functions on $S^\star X$ by

$$W_t a(z) = w_{t,1}(z)a(\psi_{1,t}(z)) + w_{t,2}(z)a(\psi_{2,t}(z)) \ ,$$

then, for almost all $z \in S^\star X$ and hence in $L^1(S^\star X, dL)$, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T W_t a dt = \int_{S^\star X} adL \ .$$

We want to show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} |\langle A\phi_j|\phi_j\rangle| = 0 \ .$$

For a bounded operator B, we denote by $\Lambda_N(B) := \frac{1}{N}\sum_{j=1}^N \langle B\phi_j|\phi_j\rangle$. We have, using the Cauchy-Schwarz inequality:

$$\frac{1}{N}\sum_{j=1}^{N} |\langle Q\phi_j|\phi_j\rangle| \leqslant \Lambda_N(Q^\star Q) \ .$$

Denoting $A_t = U(-t)AU(t)$ and $A^T = \frac{1}{T}\int_0^T A_t dt$, we get, using the fact that $U(t)$ is unitary,

$$\frac{1}{N}\sum_{j=1}^{N} |\langle A\phi_j|\phi_j\rangle| \leqslant \Lambda_N((A^T)^\star A^T) \ .$$

Moreover, from

$$\|A^T\phi_j\|^2 = \frac{1}{T^2}\int_{[0,T]^2} \langle A^\star A_{s-t}\phi_j|\phi_j\rangle dsdt \ ,$$

we get

$$\Lambda_N((A^\star)^T A^T) = \frac{1}{T^2}\int_{[0,T]^2} \Lambda_N(A^\star A_{s-t})dsdt \ . \tag{7.1}$$

Using ergodicity and given $\varepsilon > 0$, we can choose $T > 0$ so that

$$\|\frac{1}{T^2}\int_{[0,T]^2} W_{t-s}adsdt\|_{L^1(dL)} \leqslant \varepsilon \ .$$

From the decomposition

$$A_\tau = V_1(-t)AV_1(t) + V_2(-t)AV_2(t) + V_1(-t)AV_2(t) + V_2(-t)AV_1(t) ,$$

and, using the Assumption (3), we get:

$$\lim_{N\to\infty} \Lambda_N(A^\star A_\tau) = \int_{S^\star X} aW_\tau adL .$$

Applying Lebesgue dominated convergence Theorem to Equation (7.1), we get $\lim_{N\to\infty} \Lambda_N((A^\star)^T A^T) \leqslant \varepsilon$.

8. An example: Gluing together two flat disks

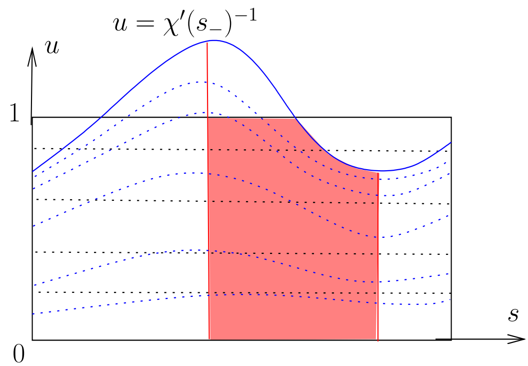

Figure 8.1. Poincaré section

Let us consider two unit Euclidian disks D_+ and D_- and a diffeomorphism $\chi : \partial D_- \to \partial D_+$ so that $\chi''(s) \neq 1$ except for a finite number of values of s. Gluing together D_+ and D_- along their boundaries using χ gives a topological manifold homeomorphic to S^2 with a metric g_χ which is flat outside the equator and discontinuous on the equator except at a finite number of points.

Let us first describe the Poincaré section: as in Section 4.2, we define $Y_\pm \subset T^\star Z$ with $Z = \mathbb{R}/2\pi\mathbb{Z}$ is the boundary of the disk D_+ parametrized by the arc length s. Then $Y_+ = \{(s,u) | s \in Z, |u| < 1\}$ with the symplectic structure $\omega = du \wedge ds$. Using the map χ and his extension Ξ to the cotangent bundle of D_-, we get $Y_- = \Xi(Z \times]-1,+1[)$ or more explicitly

$$Y_- = \{(s,u) | s \in Z, |u| < \psi(s)\}$$

with $\psi(s) = 1/\chi'(\chi^{-1}(s))$.

The Poincaré maps $P_\pm : Y_\pm \to Y_\pm$ are integrable, they preserve the foliations $\mathcal{F}_\pm$ of $Y_\pm$ defined by

- $\mathcal{F}_+ = \{L^+_\alpha | \ |\alpha| < 1\}$ with $L^+_\alpha := Z \times \{\alpha\}$ on which P_+ acts as a rotation of angle $\rho_+(\alpha) = 2\arccos(\alpha)$ satisfying $\rho'_+ < 0$
- $\mathcal{F}_- = \{L^-_\beta | \ |\beta| < 1\}$ with $L^-_\beta := \{(s, \beta\psi(s))\}$ on which P_- acts as a rotation of angle $\rho_-(\beta)$ where $\rho'_- < 0$.

We have $\psi'(s) = \eta(s)\chi''(\chi^{-1}(s)))$ where η does not vanish. This implies that the foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ are transverse in $Y_+ \cap Y_-$ outside a finite number of segments $I_j := \{s_j\} \times] - \min(1, \psi(s)), \min(1, \psi(s))[$ with $0 \leqslant s_1 < s_2 < \cdots < s_N < 2\pi$.

Our main result is:

Theorem 8.1. — *The geodesic flow on the 2-sphere (S^2, g_χ) has two ergodic components corresponding in the Poincaré sections to $u > 0$ and to $u < 0$.*

Proof. — Following the result of Section 4.2, we have to consider a subset A_0 of $Y = (Y_+ \cup Y_-) \cap \{u > 0\}$ which is invariant by P_+ and by P_-, meaning that $P_+(A_0 \cap Y_+) \equiv A_0 \cap Y_+$ and $P_-(A_0 \cap Y_-) \equiv A_0 \cap Y_-$ where $B \equiv C$ means that the symmetric difference $(B \setminus C) \cup (C \setminus B)$ has measure 0. We can replace A_0 by the intersection A of the images of A by all words in P_+ and P_-. Then $A \equiv A_0$ and is invariant by P_+ and P_-. We want to prove that A or $Y \setminus A$ has measure 0. Let A_i be the intersection of A with the leaves L^α_+ on which the rotation $\rho_+(\alpha)/2\pi$ is irrational. Then $A_i \equiv A$ and $A_i \cap L^\alpha_+$ is measurable and invariant by the rotation $\rho_+(\alpha)$. Hence the measure of $A_i \cap L^\alpha_+$ is 0 or 2π by the ergodicity of the irrational rotations of the circle. From this we get that $A \cap Y_+$ is equivalent to a set foliated by $\mathcal{F}_+$. Similarly $A \cap Y_-$ is equivalent to a set foliated by $\mathcal{F}_-$. Let us consider now the set $A_j := A \cap D_j$ with $D_j := \{(s, u) | \ s_j < s < s_{j+1}, 0 < u < \max(1, \psi(s))$. In D_j, both foliations are transverse. This implies that A_j or $D_j \setminus A_j$ is of measure 0: using smooth coordinates (x, y) in D_j so that the two foliations correspond respectively to $x = \text{const}$ and $y = \text{const}$, the indicator function of A is equivalent to a function depending on x only and to a function depending of y only, hence is equivalent to a constant 0 or 1. If A_j is of measure 0, then $A \cap Y_+$ is of measure 0 as being foliated by $\mathcal{F}_+$, similarly for $A \cap Y_-$. The conclusion follows: all A_j are of measure 0 or all $D_j \setminus A_j$ are of measure 0. □

If we want to apply Theorem 5.3, we have to take into account the fact that there are two ergodic components, they are equivalent by the involution $J : (x, \xi) \to (x, -\xi)$ which on the quantum level is the complex conjugation $\phi \to \bar{\phi}$. The semi-classical measures associated to real eigenfunctions is invariant by J.

What about the main Assumption (4)? We claim that this assumption holds for a generic diffeomorphism χ: we have the

Lemma 8.2. — *Let us consider a word $P_\chi^\alpha = P_-^{a_1} P_+^{a_2} \cdots P_+^{a_{2l}}$ with $a_j \in \mathbb{Z} \setminus 0$. For $N \geqslant 3$ and $D \subset Y_+$ a closed domain with a smooth boundary, so that $D \subset Y_+$, let $\mathcal{A}_D^N$ be the manifold of all diffeomorphisms of class C^N of S^1 so that P_χ^α is defined in some open set V containing D. Let us denote by π the projection of Y_+ onto S^1, by W the diagonal of $S^1 \times S^1$ and, for $\chi \in \mathcal{A}_D^N$, by $\rho(\chi)$ the C^2 map from D into $S^1 \times S^1$ defined by*

$$\rho(\chi)(z) = \left(\pi(z), \pi\left(P_\chi^\alpha(z)\right)\right) \ .$$

Then the set of χ's belonging to $\mathcal{A}_D^N$ so that $\rho(\chi) \pitchfork W$ is open and dense in $\mathcal{A}^N$.

Proof. — By induction on $|\alpha|$, we can assume that we are looking only at the case where the projections on S^1 of the points of the $z-$orbit $(z, P_+ z = z_1, \cdots, z_{|ga|-1})$ are pairwise distincts.

The openess is clear.

The density follows from the transversality Theorem as stated for example in [1] and [2], page 48 (see Appendix C). We will apply Theorem C.1 with $r = 2$, $X = D$, $Y = S^1 \times S^1$ and W the diagonal of Y.

The transversal intersection of $\rho(\chi)$ with W implies that the set of z for which $\pi(z) = \pi\left(P_\chi^\alpha(z)\right)$ is a submanifold of dimension 1 of Y_+. Let us consider the evaluation map $\mathrm{ev}(\chi, z) = \left(\pi(z), \pi\left(P_\chi^\alpha(z)\right)\right.$. The differential L of ev at a point (χ_0, z_0) can be written as $L(\delta\chi, \delta z) = (0, L_1 \delta\chi) + (\delta s, L_2 \delta z)$. In order to prove the transversality it is enough to prove that L_1 is surjective. Let us restrict ourselves to variations of χ in some small neighborhood of $s_1 = \chi^{-1}(s_0)$ where $z_0 = (s_0, u_0)$. Then we have $L_2(\delta\chi) = \delta\chi(s_1)$. □

Hence we get the

Proposition 8.3. — *For any $N \geqslant 3$, the set of C^N diffeomorphisms χ's, whose set of periodic points under iterations of P_+, P_- and their inverses is of measure 0, is generic.*

This implies that Assumption (4) is satisfied for a generic χ. Hence

Theorem 8.4. — *For a generic χ, any basis of real eigenfunctions of Δ_χ is QE.*

Remark 8.5. — Unique Quantum Ergodicity is not satisfied because there are infinitely many radial eigenfunctions corresponding to the Neumann and the Dirichlet problem for radial functions in the unit Euclidian disk.

9. Further questions

The important work [9] of Dmitry Jakobson, Yuri Safarov and Alexander Strohmaier inspires us several problems:

- Can we extend the result to more general wave equations, like for example the elastic wave equation where we have to take into account the polarization of waves and the mode conversions between S- and P-waves?
- What is the deviation from the QE Theorem if the assumption (3) on recombining geodesics is not fulfilled? The example of Quantum graphs (see my paper [14])could be a starting point.
- In the case of very irregular media, physicists, in particular geophysicists, use an equation called the *radiative transfer equation* (RTE) which describes the propagation of the energy of waves in the phase space (see [3] and references therein). It is known that the solutions of the RTE, after averaging over the directions, behave for large times like the solution of a diffusion equation on the configuration space, and are hence associated to some Brownian motions. This is a kind of limit of our problem as the surface X_{sing} becomes more and more complicated. Can we say something more precise?

Appendix A. A Tauberian theorem

The following Tauberian Theorem is used in the proof of Theorem 6.2.

THEOREM A.1. — *Let μ_j be an increasing sequence of real numbers satisfying a Weyl law*

$$\#\{\mu_j \leqslant \mu\} \sim W\mu^d$$

with $W > 0$ and let ρ be a smooth non-negative Schwartz function so that $\int_{\mathbb{R}} \rho(s)ds = 1$. Let us give a bounded sequence $(a_j)_{j=1,\cdots}$ and assume that

$$\sum_{j=1}^{\infty} a_j \rho(\mu - \mu_j) = A\mu^{d-1} + o\left(\mu^{d-1}\right)$$

Then we have

$$\sum_{\lambda_j \leqslant \mu} a_j = \frac{A}{dW}\mu^d + o\left(\mu^d\right)$$

This Theorem follows for example from a simple adaptation of [6, §2].

Appendix B. A stationary phase Lemma

LEMMA B.1. — *Let us define, for $\tau \in \mathbb{R}$, $I(\tau) = \int_{\mathbb{R}^N} e^{i\tau S(x)} a(x)|dx|$ where $S : \mathbb{R}^N \to \mathbb{R}$ is C^1 and $a : \mathbb{R}^N \to \mathbb{C}$ is C^1 and compactly supported, then*

$$\lim_{\tau \to \infty} I(\tau) = \int_{\{x|dS(x)=0\}} e^{i\tau S(x)} a(x)|dx| \ .$$

Proof. — Let $\varepsilon > 0$ be given; since dS is continuous, there exists $\alpha > 0$ so that $|\{0 < \|dS\| \leqslant \alpha\} \cap \mathrm{Supp}(a)| \leqslant \varepsilon$. We choose $\phi \in C_0^1(\mathbb{R}^N, [0,1])$ so that $\phi \equiv 1$ on $\{x| \ \|dS(x)\| \geqslant \alpha\} \cap \mathrm{Supp}(a)$ and $\phi \equiv 0$ on $\{x|dS(x) = 0\}$. We have

$$I(\tau) = \int_{\{x|dS(x)=0\}} e^{i\tau S(x)} a(x)|dx| + \cdots$$
$$\cdots + \int_{\{x|0<\|dS(x)\|\leqslant\alpha\}} e^{i\tau S(x)} a(x)(1-\phi(x))|dx|$$
$$+ \int_{\mathbb{R}^N} e^{i\tau S(x)} a(x)\phi(x)|dx|$$

The second integral is bounded by $\varepsilon \sup |a|$. The third one has limit 0 as $\tau \to \infty$: we integrate by parts using the facts that the vector field $V = \mathrm{grad} S/\|dS\|^2$ is continuous on the set $dS \neq 0$ and that $V(\exp(i\tau S) = i\tau \exp(i\tau S)$. Hence the third integral is $O(1/\tau)$. □

Appendix C. the Abraham-Thom transversality Theorem

Let us give the statement of the transversality Theorem, due to René Thom, as given in [12, 1, 2]; we denote by $f \pitchfork Z$ the fact that the map $f : X \to Y$ is transverse to the sub-manifold Z of Y, i.e. for each $x \in X$ so that $z = f(x) \in Z$, we have $G_z Y = f'(x)(G_x X) + G_z Z$.

THEOREM C.1. — *Let $r \geqslant 1$ and $\mathcal{A}$, X and Y be C^r manifolds. We assume that $\mathcal{A}$ is a Banach manifold while $\dim X$ and $\dim Y$ are finite. The manifold X is assumed to be compact with a smooth boundary. Consider* *a C^r map $\rho : \mathcal{A} \to C^r(X,Y)$ and $W \subset Y$ a compact sub-manifold. The evaluation map* $\mathrm{ev} : \mathcal{A} \times X \to Y$ *is defined by* $\mathrm{ev}(a,x) = \rho(a)(x)$ *and we denote by $\mathcal{A}_W$ the set of the a's in $\mathcal{A}$ so that $\rho(a) \pitchfork W$. Then if* $r > \max(0, \dim X - \mathrm{codim} W)$, *and* $\mathrm{ev} \pitchfork W$, *then $\mathcal{A}_W$ is open and dense in $\mathcal{A}$.*

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