

Institut Fourier — Université de Grenoble I

*Actes du séminaire de*  
**Théorie spectrale  
et géométrie**

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Volume 28 (2009-2010), p. 51-62.

<[http://tsg.cedram.org/item?id=TSG\\_2009-2010\\_\\_28\\_\\_51\\_0](http://tsg.cedram.org/item?id=TSG_2009-2010__28__51_0)>

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## SPECTRAL THEORY OF TRANSLATION SURFACES : A SHORT INTRODUCTION

Luc Hillairet

ABSTRACT. — We define translation surfaces and, on these, the Laplace operator that is associated with the Euclidean (singular) metric. This Laplace operator is not essentially self-adjoint and we recall how self-adjoint extensions are chosen. There are essentially two geometrical self-adjoint extensions and we show that they actually share the same spectrum

RÉSUMÉ. — On définit les surfaces de translation et le Laplacien associé à la métrique euclidienne (avec singularités). Ce laplacien n'est pas essentiellement auto-adjoint et on rappelle la façon dont les extensions auto-adjointes sont caractérisées. Il y a deux choix naturels dont on montre que les spectres coïncident.

### 1. Introduction

Spectral geometry aims at understanding how the geometry influences the spectrum of geometrically related operators such as the Laplace operator. The more interesting the geometry is, the more interesting we expect the relations with the spectrum to be. With this respect, translation surfaces form a very natural setting for spectral geometry investigations. We refer to [14] for a survey on geometrical and dynamical properties of flat and translation surfaces and to [9] for a beautiful result that expresses this interplay between spectrum and geometry.

A major difference with usual Laplace operators on smooth Riemannian manifolds is that, on translation surfaces a choice has to be made to get a self-adjoint operator. This is well-known to people working in spectral theory on singular geometries (see [2, 8] or [10] for related problems on quantum graphs) and this problem is usually dismissed in first place by considering the so-called *Friedrichs Laplacian*.

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*Keywords:* translation surfaces, flat Laplace operator, isospectrality.

*Math. classification:* 58C40, 58J53, 30Fxx.

This short note aims first at reviewing rapidly what this lack of self-adjointness means and how Friedrichs and other extensions are characterized. Then we will show that, on translation surfaces there are other natural self-adjoint extensions that are closely related to the underlying complex structure.

Finally we will prove the following isospectral result in which  $\Delta_F$  and  $\Delta_+$  are the Friedrichs Laplacian and the self-adjoint extension defined in section 4.

**THEOREM 1.1.** — *For any  $\lambda \in \mathbf{C}^*$  we have*

$$\dim(\ker(\Delta_F - \lambda)) = \dim(\ker(\Delta_+ - \lambda)).$$

*Remark 1.1.* — The dimension of  $\dim(\ker(\Delta_+))$  can be computed using Riemann-Roch theorem.

## 2. Translation surfaces

We will implicitly assume that all our surfaces are compact. We refer to [14] for a much more complete presentation of translation surfaces.

There are several ways of defining a translation surface. A very concrete one is to consider a polygon in  $\mathbf{R}^2$  (not necessarily connected) with an even number of sides that can be identified pairwise by translation. A translation surface is then obtained by making these identifications.

This surface comes naturally equipped with a Euclidean metric that is defined everywhere except at the points that correspond to identified vertices of the original polygon.

Denote by  $p$  one of these points. Near  $p$  the surface is obtained by gluing together several Euclidean triangles with vertex  $p$ . This makes  $X$  locally isometric to a neighbourhood of the tip of a Euclidean cone of some angle  $\alpha_p$ .

Since all the identifications are by translation and the original polygon has trivial holonomy, it follows that the surface also has trivial holonomy so that the angles of the conical points necessarily are multiples of  $2\pi$ . When the conical angle is exactly  $2\pi$ , the flat metric can be smoothly extended at  $p$ .

*Notation 2.1.* — We denote by  $P$  the set of conical points with angle different than  $2\pi$  and set  $X_0 = X \setminus P$ . At a singularity  $p$  we will denote the total angle of the conical singularity by  $\alpha_p$  and by  $k_p$  the integer such that  $\alpha_p = 2(k_p + 1)\pi$ . Near a conical point  $p$ , we will use polar coordinates

$(r, \theta) \in (0, r_0) \times \mathbf{R}/\alpha_p \mathbf{Z}$  where  $r_0$  is small enough so that  $\{r \leq r_0\}$  with the metric  $dr^2 + r^2 d\theta^2$  embeds isometrically into  $X$ .

A surface  $X$  than can be written  $X = X_0 \cup P$  and endowed with a (singular) metric that is Euclidean on  $X_0$  and locally isometric to a neighbourhood of the tip of a Euclidean cone near each point of  $P$  is called a Euclidean surface with conical singularities (E.C.S.C.) (see [13, 8]). An alternative definition is to say that a translation surface is a E.C.S.C. that has trivial holonomy. Observe however that the condition to have trivial holonomy is strictly more restrictive than just asking that the conical points have angle that are multiples of  $2\pi$ .

Another way of defining a translation surface is by starting from a Riemann surface  $X$  and by choosing a holomorphic one form  $\omega$ . Near points where  $\omega$  doesn't vanish it is possible to find a local holomorphic coordinate such that  $\omega = dz$ . Near a zero of order  $k$ , one can find a local coordinate so that  $\omega = (k+1)\zeta^k d\zeta$ . In both cases, we call such a local coordinate a local distinguished (holomorphic) coordinate.

We denote by  $P$  the set of zeros of  $\omega$  and by  $X_0 := X \setminus P$  and we observe that the metric  $|\omega|^2$  is flat with conical singularities thus giving  $M$  the structure of a E.S.C.S.. On  $X_0$  the set of local distinguished coordinates provides us with an atlas whose transition functions are (by definition) translations so that the holonomy is trivial and  $X$  indeed is a translation surface.

*Remark 2.1.* — The flat metric with conical singularity defines  $\omega$  only up to phase factor (since  $\exp(i\theta)\omega$  defines the same metric as  $\omega$ ).

### 3. The Laplace operator on a translation surface

#### 3.1. The Friedrichs extension

In this section we actually do not use the fact that we are dealing with a translation surface. Everything we will say actually holds true on a general E.S.C.S. We refer to [8, 6] for a more complete treatment of self-adjoint extensions of the flat Laplace operator on a E.C.S.C. The reader who is not familiar with operators and quadratic forms with domain as well as with Von Neumann theory of self-adjoint extensions will find in [12, 11, 1] (among several other good books) good references on these topics.

*Notation 3.1.* — (1) The translation surface  $X$  comes naturally equipped with a Euclidean metric with conical singularities. We denote

by  $dx$  the Euclidean area element that is associated with it and by  $L^2(X, dx)$  the associated Hilbert space of square-integrable functions. The scalar product on  $L^2(X, dx)$  is :

$$\langle u, v \rangle = \int_X u(x) \overline{v(x)} dx,$$

and  $\|\cdot\|$  is the associated norm. Near conical points we have  $dx = r dr d\theta$  in the corresponding polar coordinates.

- (2) We denote by  $\mathcal{D}(X)$  the set of smooth functions with support in  $X_0$ .
- (3) We denote by  $\nabla$  the Euclidean gradient on  $X_0$  and by  $\Delta$  the Euclidean Laplace operator on  $X_0$ . Both are well defined on functions in  $\mathcal{D}$ . The graph norm associated with  $\Delta$  is

$$\forall u \in \mathcal{D}, \|u\|_{\Delta}^2 = \|u\|^2 + \|\Delta u\|^2.$$

- (4) We also define on  $\mathcal{D}$  the following quadratic form (Dirichlet energy form) :

$$\forall u \in \mathcal{D}, q(u) = \int_X |\nabla u|^2 dx.$$

We will denote by  $\mathfrak{q}$  the polarization of  $q$ . The graph norm associated with  $q$  is

$$\forall u \in \mathcal{D}, \|u\|_q^2 = \|u\|^2 + q(u).$$

A straightforward integration by parts proves that for any  $u, v \in \mathcal{D}$ ,

$$\langle \Delta u, v \rangle = \mathfrak{q}(u, v) = \langle u, \Delta v \rangle.$$

This proves that  $\Delta$  is symmetric on the domain  $\mathcal{D}$  and that  $q$  is the quadratic form associated with  $\Delta$ . It follows that  $\Delta$  and  $q$  are closable. We will still denote by  $\Delta$  and  $q$  the operator obtained by taking the closure. By definition we thus have that  $\text{dom}(\Delta)$  is the closure of  $\mathcal{D}$  with respect to  $\|\cdot\|_{\Delta}$  and that  $\text{dom}(q)$  is the closure of  $\mathcal{D}$  with respect to  $\|\cdot\|_q$ .

We will denote by  $\Delta_F$  the unique self-adjoint operator that is associated with the closed quadratic form  $q$ . Of course,  $\Delta_F$  is a self-adjoint extension of  $\Delta$  which is known as the Friedrichs extension (compare the construction with theorem X.23 of [11]).

We define the scale of Sobolev spaces associated with the non-negative operator  $\Delta_F$ . We set  $H^s(X) := \text{dom}(\Delta_F^{\frac{s}{2}})$ . With this definition we have

$$(3.1) \quad H^1(X) = \text{dom}(q).$$

Since the surface  $X$  is compact then a Rellich-type theorem (see [2] for instance) implies that  $\Delta_F$  has compact resolvent. The spectrum of  $\Delta_F$  thus consists only of eigenvalues of finite multiplicity.

### 3.2. Von Neumann theory

We recall the following formulation of Von Neumann’s classification of self-adjoint extensions.

Let  $\mathcal{H}$  be a Hilbert space and  $A$  be an unbounded symmetric operator. Define the sesquilinear form  $\mathcal{G}$  on  $\text{dom}(A^*)$  by

$$(3.2) \quad \mathcal{G}(u, v) = \langle A^*u, v \rangle - \langle u, A^*v \rangle.$$

Since  $A$  is symmetric,  $\text{dom}(A)$  is a closed subspace of  $\text{dom}(A^*)$ . Let  $\pi$  be the canonical projection from  $\text{dom}(A^*)$  onto  $\text{dom}(A^*)/\text{dom}(A)$ .

For any  $u$  and  $v$  in  $\text{dom}(A^*)$ ,  $\mathcal{G}(u, v) = 0$ . The form  $\mathcal{G}$  thus descends as a sesquilinear form on  $\text{dom}(A^*)/\text{dom}(A)$ .

For any closed subspace  $L$  in  $\text{dom}(A^*)/\text{dom}(A)$  we define  $A_L$  to be  $A^*$  restricted to  $\pi^{-1}(L)$ . By symmetry  $A_L$  always extends  $A$  and any extension of  $A$  can be written as an  $A_L$ . The following theorem classifies the self-adjoint extensions of  $A$ .

**THEOREM 3.1.** — *The sesquilinear form  $\mathcal{G}$  defines a hermitian symplectic form on  $\text{dom}(A^*)/\text{dom}(A)$  and  $A_L$  is self-adjoint if and only if  $L$  is a lagrangian subspace for  $\mathcal{G}$ .*

### 3.3. The domain of $\Delta^*$

It turns out that the presence of conical singularities prevents  $\Delta$  from being essentially self-adjoint on the domain  $\mathcal{D}$  so that other self-adjoint extensions than  $\Delta_F$  may be considered. Since Von Neumann theory relies on an explicit description of  $\text{dom}(\Delta^*)/\text{dom}(\Delta)$ , we first describe the behaviour of elements of  $\text{dom}(\Delta^*)$  near the conical points. In order to do so, we introduce the following functions :

*Notation 3.2.* — For each conical point  $p$  of angle  $\alpha_p$  we choose a cut-off function  $\rho_p(r)$  that is identically 1 if  $r$  is small enough and identically 0 if

$r \geq r_0$  and we define the following functions, for any  $l \in \mathbf{Z}$ ,

$$\begin{aligned} F_0^+(r, \theta) &= \rho_p(r) \\ F_0^-(r, \theta) &= \log(r)\rho_p(r) \\ F_{p,l}^+(r, \theta) &= \left(\frac{2|l|\pi}{\alpha_p}\right)^{-\frac{1}{2}} r^{\frac{2|l|\pi}{\alpha_p}} \rho_p(r) \exp\left(\frac{2il\pi}{\alpha_p}\theta\right) \quad l \neq 0 \\ F_{p,l}^-(r, \theta) &= \left(\frac{2|l|\pi}{\alpha_p}\right)^{-\frac{1}{2}} r^{-\frac{2|l|\pi}{\alpha_p}} \rho_p(r) \exp\left(\frac{2il\pi}{\alpha_p}\theta\right) \quad l \neq 0 \end{aligned}$$

*Remark 3.2.* — By inspection, we see that  $F_{p,l}^+$  always belong to  $L^2(X, dx)$  whereas  $F_{p,l}^-$  belongs to  $L^2(X, dx)$  if and only if  $\frac{2|l|\pi}{\alpha_p} < 1$ .

Of course, if  $\alpha_p = 2(k+1)\pi$  the expressions can be written in a simpler way but we want to emphasize here that the following proposition is true on any E.C.S.C.

**PROPOSITION 3.3.** — *For any  $p \in P$ , define  $N_p$  to be the greatest integer  $N$  such that  $N < \frac{\alpha_p}{2\pi} + 1$ . Let  $u \in \text{dom}(\Delta^*)$  then there exists  $u_0 \in \text{dom}(\Delta)$  and, for any conical point  $p$ , there exists a collection  $(a_{p,l}^\pm)_{-N_p \leq l \leq N_p}$  such that the following holds.*

$$(3.3) \quad u = u_0 + \sum_{p \in P} \sum_{l=-N_p}^{l=N_p} a_{p,l}^+ F_{p,l}^+ + a_{p,l}^- F_{p,l}^-.$$

*Proof.* — The proof relies on the following facts. First the function  $w := (1 - \sum_p \rho_p)u$  lives on the surface  $X_\varepsilon$  where we have cut out a small ball of radius  $\varepsilon$  near each conical point. We can see, by inspection that  $w$  actually is in the domain of the Dirichlet flat Laplacian on  $X_\varepsilon$ . It follows that  $w$  is in  $\text{dom}(\Delta)$  on  $X$ . It remains to study each  $v_p := \rho_p u$ . Each of these function may be seen as living on the infinite cone of angle  $\alpha_p$ . As such it is a function of  $\text{dom}(\Delta_p^*)$  where we have denoted by  $\Delta_p$  the flat laplacian on the infinite cone. Separating variables, we are led to study the following one dimensional operators on the half-line :

$$P_\nu(u) = -\frac{1}{r}(ru')' + \frac{\nu^2}{r^2}u,$$

where  $\nu = \frac{2l\pi}{\alpha_p}$  and the reference Hilbert space is  $L^2((0, \infty), r dr)$ . These operators are essentially selfadjoint if  $\nu^2 \geq 1$  so that the sum over the corresponding modes actually is in  $\text{dom}(\Delta)$ . For the remaining modes, using Bessel functions we derive the given asymptotic expansion. (See also the appendix to section X.1 of [11])  $\square$

We set  $N = \sum_{p \in P} 2N_p + 1$  and we order the conical points and define  $A^\pm(u)$  to be the vector of  $\mathbf{C}^N$  collecting all the coefficients  $a_{p,l}^\pm$  by

putting first all the coefficients corresponding to  $p_1$  then those corresponding to  $p_2$  etc... We thus define a linear isomorphism  $A = (A^+, A^-)$  from  $\text{dom}(\Delta^*)/\text{dom}(\Delta)$  onto  $\mathbf{C}^N \times \mathbf{C}^N$ .

On  $\mathbf{C}^N$  we define the canonical scalar product  $\langle \cdot, \cdot \rangle$ ; and on  $\mathbf{C}^N \times \mathbf{C}^N$ , the canonical hermitian symplectic form  $\Omega$  by

$$\Omega((A^+, A^-), (B^+, B^-)) = \langle A^+, B^- \rangle - \langle A^-, B^+ \rangle.$$

We then obtain.

PROPOSITION 3.4. — *For any two elements  $u, v$  in  $\text{dom}(\Delta^*)$ , we have*

$$\mathcal{G}(u, v) = -2 \cdot \Omega(A \circ \pi u, A \circ \pi v).$$

For any subspace  $L \subset \mathbf{C}^N \times \mathbf{C}^N$  which is lagrangian with respect to  $\Omega, \Delta^*$  restricted to  $(A \circ \pi)^{-1}(L)$  is a self-adjoint extension of  $\Delta$  that we denote  $\Delta_L$ .

*Proof.* — For  $u$  and  $v$  in  $\text{dom}(\Delta^*)$ , write  $u = u_s + u_0$  (resp.  $v = v_s + v_0$ ) where  $u_0$  is the function defined in proposition 3.3 and  $u_s := u - u_0$  is the singular part. By symmetry of  $\Delta$ , we have

$$\mathcal{G}(u, v) = \mathcal{G}(u_s, v_s),$$

and the latter is computed directly using the followings integrals :

$$I_{\mu, \nu} = \int_0^\infty \left[ \frac{1}{r} (r(r^\mu \rho'(r)))' r^{-\nu} \rho(r) - r^\mu \rho(r) \frac{1}{r} (r(r^{-\nu} \rho'(r)))' \right] r dr.$$

When  $|\mu|, |\nu| < 1$  these integrals are finite and vanish except if  $\mu = \nu$  for which we have  $I_{\nu\nu} = -2\nu$ . The remaining statement is a direct application of theorem 3.1 □

We can now identify the lagrangian subspace  $L_F$  corresponding to the Friedrichs extension.

PROPOSITION 3.5. — *We have*

$$L_F = \{(A_+, A_-) \mid A_- = 0\}.$$

*Proof.* — Since  $\text{dom}(\Delta_F) \subset \text{dom}(q)$ , all the elements of  $\text{dom}(\Delta_F)$  must have a gradient which is square-integrable. This implies that all the coefficients  $a_{p,l}^-$  vanish. Since this defines a lagrangian subspace we have the proposition. (Compare with the definition of [2] using Hankel transform). □

As we have already said, everything in this section applies equally well to E.S.C.S. In the next section we prove that, on translation surfaces there are two other natural self-adjoint extensions.

#### 4. Cauchy-Riemann operators and DtN-Isospectrality

Since the holonomy is trivial on a translation surface, we can define directions in a consistent way. Choosing two orthonormal directions, we can define two operators  $\partial_x$  and  $\partial_y$  that are perfectly well-defined on functions of  $\mathcal{D}$ . We normalize these operators in such a way that there exists a coordinate patch on which  $\omega = dx + idy$  and  $dx(\partial_x) = 1$  and  $dy(\partial_y) = 1$ .

Using this we define the operators  $D_{\pm}$  on  $\mathcal{D}$  by :

$$(4.1) \quad D_+ = i\partial_x + \partial_y = 2i\partial_z,$$

$$(4.2) \quad D_- = i\partial_x - \partial_y = 2i\partial_{\bar{z}}.$$

For  $u \in \mathcal{D}$  we have  $\|D_{\pm}u\|^2 = q(u)$  so that  $D_{\pm}$  are closable and the domain of the closure is  $\text{dom}(q) = H^1(X)$ .

*Remark 4.1.* — It should be pointed out that the existence of conical points implies that these operators do not commute with  $\Delta_F$  (as it would be the case in the plane or on a torus). This implies that the spectrum of a translation surface cannot be computed by separating variables. It also implies that there is some function  $u$  in  $\text{dom}(\Delta_F)$  such that  $\partial_x^2$  (in the distributional sense) isn't  $L^2$ . In [3] the discrepancy between Sobolev spaces associated to the Friedrichs extension and Sobolev spaces defined by iterated regularity is studied in much greater detail.

Since for  $u \in \mathcal{D}$  we have

$$\begin{aligned} D_{\pm}^* D_{\pm} u &= D_{\mp} D_{\pm} u = \Delta u, \\ D_{\pm} D_{\pm}^* u &= D_{\pm} D_{\mp} u = \Delta u, \end{aligned}$$

we obtain four natural reasonable self-adjoint extension of  $\Delta$ .

The following lemma actually says that two of these are the Friedrichs extension.

LEMMA 4.1. — *We have*

$$D_{\pm}^* D_{\pm} = \Delta_F.$$

*Proof.* — This follows from the fact that  $\text{dom}(D_{\pm}^* D_{\pm}) \subset \text{dom}(D_{\pm}) = H^1(X)$ , and the Friedrichs extension is the only extension of  $\Delta$  whose domain is a subset of  $H^1(X)$ .  $\square$

*Notation 4.2.* — We set  $D_{\pm} D_{\pm}^* := \Delta_{\pm}$  and define  $L_{\pm}$  to be the corresponding lagrangian subspaces of  $\mathbf{C}^N \times \mathbf{C}^N$ .

*Remark 4.2.* — Using (4.1) and (4.2), we see that if  $F$  is  $L^2$  and holomorphic in  $X_0$ , then  $f \in \text{dom}(D_-^*)$  and  $D_-^* f = 0$ . The same is true for  $D_+^*$  and antiholomorphic  $L^2$  functions. (See also [3])

We can also remark, that in the ball centered at  $p$  of radius  $\varepsilon$  we have

$$(4.3) \quad \begin{cases} F_{p,l}^+(r, \theta) = c_{p,l}^+ \zeta^l & l > 0, \\ F_{p,l}^-(r, \theta) = c_{p,l}^- \bar{\zeta}^{-l} & l > 0, \\ F_{p,l}^-(r, \theta) = c_{p,l}^- \zeta^l & l < 0, \\ F_{p,l}^+(r, \theta) = c_{p,l}^+ \bar{\zeta}^{-l} & l < 0; \end{cases}$$

where the  $c_{p,l}^\pm$  are normalizing constants.

This allows to prove the following proposition.

**PROPOSITION 4.3.** — *An element  $u \in \text{dom}(\Delta^*)$  belongs to  $\text{dom}(\Delta_-)$  if and only if*

$$(4.4) \quad \begin{cases} a_{p,l}^- = 0 & \text{if } l \geq 0 \\ a_{p,l}^+ = 0 & \text{if } l < 0. \end{cases}$$

*An element  $u \in \text{dom}(\Delta^*)$  belongs to  $\text{dom}(\Delta_+)$  if and only if  $\bar{u} \in \text{dom}(\Delta_-)$ , or, alternatively, if the following set of conditions is satisfied :*

$$(4.5) \quad \begin{cases} a_{p,l}^- = 0 & \text{if } l \leq 0 \\ a_{p,l}^+ = 0 & \text{if } l > 0. \end{cases}$$

*Proof.* — Using remark 4.2, we see that if  $u$  is holomorphic near  $p$  then it belongs to  $\text{dom}(\Delta_-)$ . This implies that if the coefficients of  $\bar{\zeta}^l$  in the expansion (3.3) vanish then  $u \in \text{dom}(\Delta_+)$ . Comparing with (4.3), it gives the inclusion of  $\text{dom}(\Delta_-)$  into the set defined by the equations (4.4). Since both these subspaces are lagrangian we obtain the equality. The same reasoning with antiholomorphic functions yields the result for  $\Delta_+$ . □

**COROLLARY 4.4.** — *The operators  $\Delta_+$  and  $\Delta_-$  are isospectral.*

*Proof.* — Indeed, the complex conjugation is invertible and intertwines both operators. □

### 4.1. DTN Isospectrality and commutators

In this last section we will derive two applications using these operators  $\Delta_+$ . The first one is some kind of isospectrality that we define now.

DEFINITION 4.3 (Dirichlet-to-Neumann isospectrality). — *Let  $A_1$  and  $A_2$  be two self-adjoint operators with compact resolvent such that*

- (1)  $A_1$  and  $A_2$  are two self-adjoint extensions of the same operator  $A_0$ .
- (2) For any  $\lambda \neq 0$ , the eigenspaces  $\ker(A_1 - \lambda)$  and  $\ker(A_2 - \lambda)$  have the same dimension.

*Then we will say that  $A_1$  and  $A_2$  are Dirichlet-to-Neumann isospectral. (DTN-isospectral).*

There are several examples in the literature of DTN-isospectrality and sometimes the condition  $\lambda \neq 0$  may even be dropped.

- (1) The operators  $u \mapsto -u''$  on  $[0, 1]$  with Dirichlet and Neumann boundary condition are DTN-isospectral.
- (2) So are the examples of domains with mixed Dirichlet-Neumann boundary condition of [7].
- (3) And so are operators in the class  $\mathcal{O}_N$  of [5] that correspond to the same representation (see proposition 2 of [5]).

We have the following theorem.

THEOREM 4.4. — *The operators  $\Delta_F$  and  $\Delta_+$  are DTN-isospectral.*

*Proof.* — It is actually a general fact that  $AB$  and  $BA$  have the same non-zero spectrum. The proof runs as follows. Let  $u \neq 0$  be an element of  $\ker(\Delta_F - \lambda)$  then  $u$  is characterized by  $\Delta^*u = \lambda u$  and  $u \in \text{dom}(\Delta_F)$ . Since  $\text{dom}(\Delta_F) \subset H^1(X)$  we can apply  $D_+$  to  $u$  and compute  $\Delta^*D_+u$ . For any test function  $\phi \in \mathcal{D}$  we have

$$\begin{aligned} \langle D_+u, \Delta\phi \rangle &= \langle u, D_-\Delta\phi \rangle \\ &= \langle u, \Delta D_-\phi \rangle \\ &= \langle \Delta_F u, D_-\phi \rangle \\ &= \lambda \langle u, D_-\phi \rangle \\ &= \lambda \langle D_+u, \phi \rangle. \end{aligned}$$

Thus we have  $\Delta^*D_+u = \lambda D_+u$ . It remains to show that  $D_+u$  belongs to  $\text{dom}(\Delta_+)$ . This can be seen directly by considering the action of  $D_+$  on the functions  $F_{p,l}$  or using a more abstract argument. Since  $\lambda \neq 0$   $D_+u \neq 0$  so that we have an injective linear map from  $\ker(\Delta_F - \lambda)$  into  $\ker(\Delta_+ - \lambda)$ . The same reasoning proves that  $D_+^*$  defines an injective linear map in the other direction. This yields the result.  $\square$

The second application is concerned with commutation of  $\Delta_F$  and  $D_\pm$ . As it has already been pointed out, one should take care that, on a translation surface that isn't a torus, the operators  $\partial_x$  and  $\partial_y$  do not commute.

Neither do  $\Delta_F$  and  $D_{\pm}$  commute. The following proposition points out the fact that the lack of commutation can be described by a finite rank operator.

We introduce the family of operators  $K(\lambda)$  such that

$$(\Delta_+ - \lambda)^{-1} = (I + K(\lambda))(\Delta_F - \lambda)^{-1}$$

This construction is standard in perturbation theory. In the case of changing the self-adjoint condition it is closely related to the Dirichlet-to-Neumann operator. In our case, since  $\text{dom}(\Delta^*)/\text{dom}(\Delta)$  is finite-dimensional the family  $K(\lambda)$  is finite rank. (See also [4, 6]).

**PROPOSITION 4.5.** — *For any  $\lambda$  in the resolvent set of  $\Delta_F$ , and for any  $f \in H^1(X)$  we have*

$$\left[ D_+, (\Delta_F - \lambda)^{-1} \right] f = K(\lambda) (\Delta_F - \lambda)^{-1} D_+ f.$$

*Proof.* — Denote by  $u_F$  the solution to

$$(\Delta_F - \lambda) u_F = f.$$

We apply  $D_+$  to both sides of the equation. The proof of Theorem 4.4 implies the following commutation property

$$D_+ (\Delta_F - \lambda) u_F = (\Delta_+ - \lambda) D_+ u_F.$$

We thus obtain

$$D_+ (\Delta_F - \lambda)^{-1} f = (\Delta_+ - \lambda)^{-1} D_+ f.$$

The claim follows by replacing the resolvent of  $\Delta_+$  using the definition of  $K(\lambda)$ .  $\square$

*Remark 4.5.* — Observe that  $K(\lambda)$  may be expressed using the  $S$ -matrix formalism of [6].

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