Institut Fourier — Université de Grenoble I

Actes du séminaire de

Théorie spectrale et géométrie

Sylvain MAILLOT

Some applications of Ricci flow to 3-manifolds

Volume 25 (2006-2007), p. 121-148.

<http://tsg.cedram.org/item?id=TSG_2006-2007__25__121_0>

Séminaire de théorie spectrale et géométrie
GRENOBLE
Volume 25 (2006-2007) 121-148

SOME APPLICATIONS OF RICCI FLOW TO 3-MANIFOLDS

Sylvain Maillot

1. Introduction

The purpose of this text is to describe some applications of Ricci flow to questions about the topology and geometry of 3-manifolds. The most spectacular achievement in the recent years was the proof of W. Thurston's Geometrization Conjecture by G. Perelman [39, 41, 40].

In Sections 2–6 we outline a proof of the Geometrization Conjecture which is based in part on Perelman's ideas. This is joint work with L. Bessières, G. Besson, M. Boileau, and J. Porti. Some details have already appeared in the preprint [4]; the rest will be contained in our forthcoming monograph [3]. In Section 7 we announce some results on the topological classification of (possibly noncompact) 3-manifolds with positive scalar curvature. These results follow from an extension of those discussed in Section 6, and are joint work with L. Bessières and G. Besson. Details will appear elsewhere.

The theory of Ricci flow in low dimensions relies on techniques and insights from riemannian geometry, geometric analysis, and topology. In this text, we deliberately adopt the topologist's viewpoint; thus we shall focus on the topological and geometric arguments rather than the analytic aspects. The Ricci flow is considered here as a tool to prove topological and geometric theorems. Therefore, we sometimes do not prove the strongest possible estimates on Ricci flow solutions, but rather strengthen the theorems that are used to deduce topological consequences (e.g. Theorem 4.2

Acknowledgements: partially supported by ANR project ANR-07-BLAN-0251-01.

or Proposition 5.11) by weakening their hypotheses. It should go without saying that the Ricci flow as a mathematical object is worth studying for its own sake, which justifies trying to obtain the best possible results on the analytic side.

This text is mostly intended for nonexperts. The prerequisites are basic concepts in algebraic topology and differential geometry; previous knowledge of 3-manifold theory or Ricci flow is of course useful, but not necessary. We have endeavored to make the various parts of the proof as independent from each other as possible, in order to clarify its overall structure.

Special paragraphs marked with an asterisk (*) are geared toward the experts who wish to understand the differences between the proof of the Geometrization Theorem presented here and Perelman's proof. Only in those paragraphs is some familiarity with Perelman's papers implicitly assumed.

Acknowledgments. — I would like to thank all the people who have, directly or indirectly, helped me understand Perelman's Ricci flow papers. In addition to my above-mentioned co-authors, this includes the participants of the workshops in Grenoble 2004 and München 2005, especially Bernhard Leeb, Thomas Schick, Hartmut Weiss and Jonathan Dinkelbach. I am also indebted to Bruce Kleiner and John Lott for writing their notes [30], as well as the Clay Mathematics Institute for financial support.

Numerous conversations, in particular with Thomas Delzant and Olivier Biquard, have helped shape my thoughts on the subject. I also thank the organizers of the seminars and conferences where I have had the opportunity to present this work. The present text is based in part on notes from those talks. Lastly, I would like to thank Laurent Bessières and Gérard Besson for their remarks on a preliminary version of this paper.

2. Geometrization of 3-manifolds

All 3-manifolds considered in this text are assumed to be smooth, connected, and orientable. The n-dimensional sphere (resp. n-dimensional real projective space, resp. n-dimensional torus) is denoted by S^n (resp. RP^n, resp. T^n). General references for this section are [43, 6, 7, 25, 26].

We are mostly interested in compact manifolds, and more precisely *closed* manifolds, i.e., compact manifolds whose boundary is empty. A closed 3-manifold is *spherical* if it admits a riemannian metric of constant positive sectional curvature. Equivalently, a 3-manifold is spherical if it can be obtained as a quotient of S^3 by a finite group of isometries acting freely. In particular, the universal cover of a spherical 3-manifold is diffeomorphic to S^3.

In this text, a 3-manifold H with empty boundary is called *hyperbolic* if it admits a complete riemannian metric of constant sectional curvature -1 and finite volume. Such a metric is called a *hyperbolic metric*. A hyperbolic manifold may be closed, or have finitely many ends, called *cusps*, which admit neighborhoods diffeomorphic to $T^2 \times [0, +\infty)$. The hyperbolic metric, which by Mostow rigidity is unique up to isometry, is denoted by g_{hyp}. By extension, a compact manifold M with nonempty boundary is called hyperbolic if its interior is hyperbolic. Thus in this case ∂M is a union of tori.

A 3-manifold is *Seifert fibered*, or simply *Seifert*, if it is the total space of a fiber bundle over a 2-dimensional orbifold. It is well-known that all spherical 3-manifolds are Seifert fibered. Seifert manifolds have been classified, and form a well-understood class of 3-manifolds. Among them, spherical manifolds are exactly those with finite fundamental group.

To state the Geometrization Theorem, we still need two definitions: a 3-manifold M is *irreducible* if every embedding of the 2-sphere into M can be extended to an embedding of the 3-ball. An orientable embedded surface $F \subset M$ of positive genus is called *incompressible* if the group homomorphism $\pi_1 F \to \pi_1 M$ induced by the inclusion map is injective.

The main purpose of this article is to present a proof of the following result:

THEOREM 2.1 (Perelman). — *Let M be a closed, irreducible 3-manifold. Then M is hyperbolic, Seifert fibered, or contains an embedded incompressible torus.*

Theorem 2.1, together with the Kneser-Milnor prime decomposition theorem (see e.g. [25],) the torus splitting theorem of Jaco-Shalen [27] and Johannson [28], and Thurston's hyperbolization theorem for Haken manifolds [37, 36, 29], implies Thurston's original formulation of the Geometrization Conjecture in terms of a canonical decomposition of any compact 3-manifold along spheres and tori into 'geometric' 3-manifolds admitting locally homogeneous riemannian metrics.(1)

Theorem 2.1 implies the Poincaré Conjecture: indeed, by Kneser's theorem and van Kampen's theorem, it is enough to prove this conjecture for

(1) Of the famous 'eight geometries' that uniformize those locally homogeneous 3-manifolds, six correspond to Seifert manifolds (including spherical geometry), one is hyperbolic geometry, and the last one, **Sol**, does not appear in the above statement. (From our viewpoint, **Sol** manifolds are just special 3-manifolds containing incompressible tori.)

irreducible 3-manifolds. Now if $\pi_1 M$ is trivial, then M cannot be hyperbolic or contain an incompressible torus. Hence M is Seifert. As mentioned earlier, Seifert manifolds with finite fundamental group are spherical. It follows that M is a quotient of the 3-sphere by a trivial group, i.e. the 3-sphere itself.

By a straightforward extension of the above argument, one can deduce from Theorem 2.1 the following strengthening of the Poincaré Conjecture:

Elliptization Conjecture. — Every closed 3-manifold with finite fundamental group is spherical.

When $\pi_1 M$ is infinite, it was known from [44, 33, 47, 16, 9] (see also [32] and [7, Chapter 5]) that if $\pi_1 M$ has a subgroup isomorphic to $\mathbf{Z}^2$, then it is Seifert fibered or contains an embedded incompressible torus. Thus, prior to Perelman's work, the only remaining open question was the following:

Hyperbolization Conjecture. — If M is a closed, irreducible 3-manifold whose fundamental group is infinite and does not contain a subgroup isomorphic to $\mathbf{Z}^2$, then M is hyperbolic.

This statement is also a direct consequence of Theorem 2.1.

The dichotomy finite vs infinite fundamental group will not appear directly in the proof presented here, but rather via properties of the higher homotopy groups. The connection is given by the following well-known lemma:

Lemma 2.2. — *Let M be a closed, irreducible* 3*-manifold. Then the following are equivalent:*

(i) $\pi_1 M$ *is infinite;*
(ii) $\pi_3 M$ *is trivial;*
(iii) M *is* aspherical, *i.e.* $\pi_k M$ *is trivial for all* $k \geqslant 2$.

Proof. — It follows from irreducibility of M and the Sphere Theorem that $\pi_2 M$ is trivial. Let $\tilde{M}$ be the universal cover of M. Then $\tilde{M}$ is 2-connected. By the Hurewicz Isomorphism theorem, we have $\pi_3 \tilde{M} \cong H_3 \tilde{M}$. Hence $\pi_3 M$, which is isomorphic to $\pi_3 \tilde{M}$, vanishes if and only if $\tilde{M}$ is noncompact. This proves that (i) and (ii) are equivalent.

It is immediate that (iii) implies (ii). For the converse, apply the Hurewicz theorem inductively to $\tilde{M}$. □

3. The Ricci Flow approach

Notation 3.1. — If g is a riemannian metric, we denote by $R_{\min}(g)$ the minimum of its scalar curvature, by Ric_g its Ricci tensor, and by $\mathrm{vol}(g)$ its volume.

Let M be a closed, irreducible 3-manifold. In the Ricci flow approach to geometrization, one studies solutions of the evolution equation

$$\frac{dg}{dt} = -2\mathrm{Ric}_{g(t)}, \tag{3.1}$$

called the *Ricci flow equation*, which was introduced by R. Hamilton. A solution is an *evolving metric* $\{g(t)\}_{t\in I}$, i.e. a 1-parameter family of riemannian metrics on M defined on an interval $I \subset \mathbf{R}$. In [20], Hamilton proved that for any metric g_0 on M, there exists $\varepsilon > 0$ such that Equation (3.1) has a unique solution defined on $[0, \varepsilon)$ with initial condition $g(0) = g_0$. Thus there exists $T \in (0, +\infty]$ such that $[0, T)$ is the maximal interval where the solution to (3.1) with initial condition g_0 is defined. When T is finite, one says that Ricci flow has a *singularity* at time T. Ideally, one would like to see the geometry of M appear by looking at the metric $g(t)$ when t tends to T (whether T be finite or infinite.) To understand how this works, we first consider some (very) simple examples, where the initial metric is locally homogeneous.

Example 3.2. — If g_0 has constant sectional curvature K, then the solution is given by $g(t) = (1 - 4Kt)g_0$. Thus in the spherical case, where $K > 0$, we have $T < \infty$, and as t goes to T, the manifold shrinks to a point while remaining of constant positive curvature.

By contrast, in the hyperbolic case, where $K < 0$, we have $T = \infty$ and $g(t)$ expands indefinitely, while remaining of constant negative curvature. In this case, the *hyperbolically rescaled* solution $\tilde{g}(t) := (4t)^{-1}g(t)$ converges to the metric of constant sectional curvature -1.

Example 3.3. — If M is the product of a circle with a surface of genus at least 2, and g_0 is a product metric whose second factor has constant negative curvature, then $T = \infty$; moreover, the metric is constant on the first factor, and expanding on the second factor.

In this case, $g(t)$ does not have any convergent rescaling. However, one can observe that the hyperbolically rescaled solution $\tilde{g}(t)$ defined above *collapses with bounded curvature*, i.e. has bounded curvature and injectivity radius going to 0 everywhere as t goes to $+\infty$. Manifolds admitting collapsing sequences of riemannian metrics under various curvature bounds have been studied by several authors, starting with the seminal work of Cheeger-Gromov [10, 11] (see also the surveys [38, 15], as well as the recent paper [45] and the references therein.) Thus one is led to expect that these techniques can be used to deduce topological information on M from the large-scale behavior of $\tilde{g}(t)$ in this case.

One can give similar formulae for other locally homogeneous Ricci flows (see e.g. [31].)

An (overly) optimistic program based on the previous examples would go as follows: if $\pi_1 M$ is finite, prove that any Ricci flow solution has a singularity at some finite time T, and that the spherical metric can be obtained as a limit of rescalings of $g(t)$ when t goes to T. If $\pi_1 M$ is infinite, show that Ricci flow is defined for all times, and study the long time behavior of the hyperbolically rescaled solution $\tilde{g}(t) := (4t)^{-1} g(t)$ in order to recognize the topological type of M; for instance, if M has a hyperbolic metric, then this metric ought to appear as the limit of $\tilde{g}(t)$ as t tends to infinity.

Several important results were obtained by Hamilton [20, 21, 24] in this direction, among which we state just two. When g_0 has positive Ricci curvature, then $T < \infty$, and the volume-rescaled Ricci flow $\mathrm{vol}(g(t))^{-2/3} g(t)$ converges to a round metric as $t \to T$. If $T = \infty$ and $\tilde{g}(t)$ has uniformly bounded sectional curvature, then it converges or collapses, or M contains an incompressible torus.(2)

The general case, however, is more difficult, because it sometimes happens that $T < \infty$ while the behavior of $g(t)$ as t tends to T does not allow to determine the topology of M. One possibility is the so-called *neck pinch*, where part of M looks like a thinner and thinner cylindrical neck as one approaches the singularity. This can happen even if M is irreducible (see [2] for an example where $M = S^3$); thus neck pinches may not give any useful information on the topology of M. See [22] and [12, Chapters 8 and 9] for a discussion of what was known on formation of singularities prior to Perelman's work.

A solution to this problem was found by G. Perelman, inspired by ideas of Hamilton [23]. In [41], Perelman explains how to construct a kind of generalized solution to the Ricci flow equation, which he calls *Ricci flow with δ-cutoff*. This type of solution exists for all time unless it leads to a metric that is sufficiently controlled so as to allow one to recognize the topology of the manifold. Several slightly different ways to make this construction

(2) Hamilton's original results were formulated in terms of normalized Ricci flow. We have restated them so that they fit better in our discussion.

precise are given in [30, 34, 8]. We shall work with closely related objects which we call *weak solutions* of the Ricci flow equation.

The first part of the proof of Theorem 2.1 is to prove an existence theorem for weak solutions. The proof then splits into two cases.

If $\pi_1 M$ is finite, then by Lemma 2.2, we have $\pi_3 M \neq 0$. Following [13], we use this fact to associate to any riemannian metric g a quantity called its *width* $W(g)$. This invariant can be studied via minimal surface theory. In particular, one can control the way it varies with time in a weak solution $\{g(t)\}$. This is used to prove that for some finite time t_0, the metric $g(t_0)$ belongs to a special class of metrics, called *locally canonical* metrics. This permits to recognize the topology of M.

If $\pi_1 M$ is infinite, then we need to refine the existence theorem for weak solutions. We prove that such solutions exist for all time, and enjoy additional properties. Then we study the long time behavior of the corresponding hyperbolically rescaled solutions to deduce topological consequences.

In Section 4, we give a precise definition of weak solutions, state an existence result, and explain how to deduce the Elliptization Conjecture. The case where $\pi_1 M$ is infinite is tackled in Section 5. In Section 6, we discuss in more detail the construction of weak solutions.

4. Canonical neighborhoods, weak solutions and elliptization

4.1. Canonical neighborhoods

Let ε be a positive number. The *standard ε-neck* is the riemannian product $N_\varepsilon := S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1})$, where the S^2 factor is round of scalar curvature 1. Its metric is denoted by g_0^ε.

Definition 4.1. — *Let ε be a positive number, (M, g) be a riemannian 3-manifold and x be a point of M. A neighborhood $U \subset M$ of x is called a* weak ε-canonical neighborhood *if one of the following holds:*

(i) *There exist a number $\lambda > 0$ and a $\mathcal{C}^{[\varepsilon^{-1}]+1}$-diffeomorphism $f : (N_\varepsilon, *) \to (U, x)$ (where N_ε is the standard ε-neck, and $*$ is a basepoint in $S^2 \times \{0\}$) such that the $\mathcal{C}^{[\varepsilon^{-1}]+1}$-norm of the tensor $\phi^*(\lambda g) - g_0^\varepsilon$ is less than ε everywhere. In this case, we say that U is an* ε-neck *centered at x.*

(ii) *U is the union of two sets V, W such that $x \in V$, V is a closed topological 3-ball, $\overline{W} \cap V = \partial V$, and W is an ε-neck. Such a U is called an* ε-cap *centered at x.*

(iii) *U is a spherical 3-manifold (hence $U = M$ since M is connected).*

This notion is useful for topological purposes because of the following result (cf. [34, Proposition A.25]):

THEOREM 4.2. — *Let (M, g) be a closed riemannian 3-manifold. If every point of (M, g) has a weak 10^{-2}-canonical neighborhood, then M is spherical or diffeomorphic to $S^2 \times S^1$.*

Sketch of proof. — If some point has a neighborhood of the third type, there is nothing to prove. Otherwise, there are two cases: if all points are centers of necks, then by piecing together those necks, one obtains a fibration of M by 2-spheres. Since M is orientable, it must be diffeomorphic to $S^2 \times S^1$. If some point is the center of a cap, then one shows that M is either the union of two caps, or the union of two caps and a chain of necks connecting them. In either case, M is diffeomorphic to S^3. □

Theorem 4.2 motivates the following definition:

DEFINITION 4.3. — *A riemannian metric g on a 3-manifold M is* locally canonical *if every point of (M, g) has a weak 10^{-2}-canonical neighborhood.*

Remark 4.4. — (*) Definition 4.1 is weaker than Perelman's in several respects. Perelman's definition includes additional geometric information such as estimates on the curvature or its derivatives, which are not needed for Theorem 4.2. He also considers caps that are diffeomorphic to the punctured real projective 3-space instead of the 3-ball. We do not need to do this because we shall soon restrict attention to 3-manifolds without embedding projective planes.

Case (iii) of Definition 4.1 may appear somewhat artificial. A more natural definition would include some geometric information related to the formation of singularities of the Ricci flow. However, this requires splitting case (iii) into two subcases, which creates complications irrelevant to the topological applications.

4.2. Weak solutions

Until the end of this section, we suppose that M is closed and irreducible. We also assume that M is *RP^2-free*, i.e. does not contain any submanifold diffeomorphic to RP^2. This is not much of a restriction because the only closed, irreducible 3-manifold that does contain an embedded copy of RP^2 is RP^3, which is a spherical manifold.

The next goal is to give the formal definition of weak solutions and state an existence theorem. To motivate this, we first give a very broad outline of the construction, which will be developped in Section 6.

As we already explained, one of the main difficulties in the Ricci flow approach to geometrization is that singularities unrelated to the topology of M may appear. Using maximum principle arguments, one shows that singularities in a 3-dimensional Ricci flow can only occur when the scalar curvature tends to $+\infty$ somewhere. One of Perelman's major breakthroughs was to give a precise local description of the geometry at points of large scalar curvature. In particular, he showed that those points have weak canonical neighborhoods.

To solve the problem of singularities, we fix a large number Θ, which plays the role of a curvature threshold. As long as the maximum of the scalar curvature is less than Θ, Ricci flow is defined. If it reaches Θ at some time t_0, then there are two possibilities: if the minimum of the scalar curvature of the time-t_0 metric is large enough, then we shall see that this metric is locally canonical, so that Theorem 4.2 enables us to recognize the topology of M.

Otherwise, we modify g_0 so that the maximum of the scalar curvature of the new metric, denoted by $g_+(t_0)$, is at most $\Theta/2$. This modification is called *metric surgery*. It consists in replacing some ε-caps by a special type of ε-caps called *almost standard caps*. Then we start the Ricci flow again, using $g_+(t_0)$ as new initial metric. This procedure is repeated as many times as necessary. The main difficulty is to choose Θ and do metric surgery in such a way that the construction can indeed by iterated.

We now come to the formal definitions. Let M be a 3-manifold.

An *evolving metric* on M defined on an interval $[a, b]$ is a map $t \mapsto g(t)$ from $[a, b]$ to the space of smooth riemannian metrics on M. It is *piecewise* $\mathcal{C}^1$ if there is a finite subdivision $a = t_0 < t_1 < \cdots < t_p = b$ such that the map defined on $[t_i, t_{i+1}]$ by sending t_i to $g_+(t_i)$ and t to $g(t)$ for all $t \in (t_i, t_{i+1}]$ is $\mathcal{C}^1$-smooth.

We often view an evolving metric $g(\cdot)$ as a 1-parameter family of metrics indexed by the interval $[a, b]$; thus we use the notation $\{g(t)\}_{t\in[a,b]}$. For $t \in [a, b]$, we say that t is *regular* if $g(\cdot)$ is $\mathcal{C}^1$-smooth in a neighborhood of t. Otherwise t is called *singular*. By definition, the set of singular times is finite. If $t \in (a, b)$ is a singular time, then it follows from the definition that the map $g(\cdot)$ is continuous from the left at t, and has a limit from the right, denoted by $g_+(t)$.

There are similar definitions where the domain of definition $[a, b]$ is replaced by an open or a half-open interval I. When I has infinite length, the set of singular times is a discrete subset of the real line, but it may be infinite.

DEFINITION 4.5. — *Let $I \subset \mathbf{R}$ be an interval. A piecewise $\mathcal{C}^1$ evolving metric $t \mapsto g(t)$ on M defined on I is said to be a* weak solution *of the Ricci flow equation (3.1) if*

(i) *Equation (3.1) is satisfied at all regular times.*
(ii) *For every singular time $t \in I$ one has*
  (a) $R_{\min}(g_+(t)) \geqslant R_{\min}(g(t))$, *and*
  (b) $g_+(t) \leqslant g(t)$.

We now state the main technical result on weak solutions needed to prove the Elliptization Conjecture (cf. [41, Proposition 5.1]):

THEOREM 4.6 (Existence of weak solutions). — *Let M be a closed, irreducible, RP^2-free 3-manifold. For every $T > 0$ and every riemannian metric g_0 on M, there exists $T' \in (0, T]$ and a weak solution $\{g(t)\}$ on M, defined on $[0, T']$, with initial condition $g(0) = g_0$, and such that either*

(i) $T' = T$, *or*
(ii) $T' < T$ *and $g(T')$ is locally canonical.*

Remark 4.7. — The statement of Theorem 4.6 is slightly weaker than one might expect, since in case (i) it is not claimed that the solution exists for all time. A stronger statement is in fact true (cf. Theorem 5.7 below), but its proof is more involved.

Remark 4.8. — (*) There are a few differences between weak solutions and Perelman's Ricci flow with δ-cutoff. One obvious difference is that a weak solution is an evolving metric on a fixed manifold rather than an evolving manifold. This simplification is made possible by the extra topological assumptions on M.

Another, perhaps more significant difference is that surgery occurs *before* the Ricci flow becomes singular, rather than *at* the singular time. Our construction is in this respect closer in spirit to the surgery process envisioned by Hamilton [23]. This point is discussed in more detail in Section 6.

4.3. Proof of elliptization

To deduce the elliptization part of Theorem 2.1, all we need is the following result (cf. [40, 13]:)

THEOREM 4.9. — *Let M be a closed, irreducible, RP^2-free 3-manifold with finite fundamental group. For each riemannian metric g_0 on M there is a number $T(g_0) > 0$ such that if $\{g(t)\}$ on M is a weak solution defined on some interval $[0, T]$ and with initial condition $g(0) = g_0$, then $T < T(g_0)$.*

Indeed, assume that Theorems 4.2, 4.6, and 4.9 have been proved. Let M be a closed, irreducible, RP^2-free 3-manifold such that $\pi_1 M$ is finite, and let g_0 be an arbitrary metric on M. Theorem 4.9 provides a positive number $T(g_0) > 0$ such that no weak solution with initial condition g_0 can exist up to time $T(g_0)$. Let $T' > 0$ and $\{g(t)\}_{0 \leqslant t < T'}$ be given by Theorem 4.6 applied with $T = T(g_0)$. Then Case (ii) of the conclusion of that theorem must hold. This implies that M has a locally canonical metric. Applying Theorem 4.2 and noting that $S^2 \times S^1$ is not irreducible, we conclude that M is spherical.

Let us sketch the proof of Theorem 4.9. Let M be a 3-manifold satisfying the hypotheses and g_0 be a riemannian metric on M. First we show, following Colding-Minicozzi [13], that there is a constant T such that there exists no Ricci flow solution on M defined on $[0, T]$, with initial condition $g(0) = g_0$. It will then be straightforward to extend the argument to prove that there can in fact exist no weak solution defined on the same interval.

Let Ω be the space of smooth maps $f : S^2 \times [0, 1] \to M$ such that $f(S^2 \times \{0\})$ and $f(S^2 \times \{1\})$ are points. Since $\pi_1 M$ is finite, Lemma 2.2 implies that $\pi_3 M \neq 0$. It follows that there exists $f_0 \in \Omega$ which is not homotopic to a constant map. Let ξ be the homotopy class of f_0. We set

$$W(g) := \inf_{f \in \xi} \max_{s \in [0,1]} E(f(\cdot, s)),$$

where E denotes the energy

$$E(f(\cdot, s)) = \frac{1}{2} \int_{S^2} |\nabla_x f(x, s)|^2 \, dx.$$

Let $\{g(t)\}_{t \in [0,T]}$ be a Ricci flow solution such that $g(0) = g_0$. The function $t \mapsto W(g(t))$ is continuous. Colding and Minicozzi [13, 14] prove that there exists a constant $C \geqslant 0$ depending only on g_0 such that:

$$\frac{d}{dt} W(g(t)) \leqslant -4\pi + \frac{3}{4(t + C)} W(g(t)), \tag{4.1}$$

where $\frac{d}{dt} W(g(t))$ is to be interpreted as

$$\limsup_{h > 0, h \to 0} \frac{W(g(t + h)) - W(g(t))}{h}$$

in case $W(g(t))$ is not differentiable at t in the traditional sense.

Since the right-hand side is not integrable and $W(g(t))$ cannot become negative, this easily implies an upper bound on T depending only on C, hence on g_0.

If we replace $\{g(t)\}_{t\in[0,T]}$ by a weak solution, then $W(g(t))$ need not be continuous in t. However, condition (ii)(b) in the definition of a weak solution immediately implies that

$$W(g_+(t)) \leqslant W(g(t)) \tag{4.2}$$

if t is a singular time. One can show that the inequality (4.1) still holds at regular times. This is enough to conclude that the previous a priori upper bound on the existence time of Ricci flow also applies to weak solutions.

Remark 4.10. — (*) In the above argument, property (ii)(b) of the definition of weak solutions, which amounts to the fact that surgery does not increase distances, was used in a crucial way. This part of the proof has been simplified: indeed, when one uses Perelman's surgery construction, the pre-surgery metric is not defined on the whole manifold, which makes the proof of (4.2) more complicated (cf. [5]).

Remark 4.11. — (*) Alternatively, one can replace $W(g)$ by Perelman's invariant $A(\alpha, g)$ introduced in [40] and follow [34] for the derivation of the inequality analogous to (4.1). Working with weak solutions in our sense as opposed to Ricci flow with δ-cutoff leads to a similar simplification.

5. The aspherical case

In this section, we outline the proof of the other 'half' of the geometrization theorem 2.1, where $\pi_1 M$ is assumed to be infinite. Subsection 5.1 is devoted to background material on Haken manifolds, graph manifolds and the JSJ-decomposition. In Subsection 5.2, we state Theorem 5.7, which contains all the results on the Ricci flow that are needed in the proof. In Subsection 5.3, we explain how to use this result to reach the desired conclusion.

5.1. Some 3-manifold theory

A *Haken* 3-manifold is a compact, irreducible 3-manifold that contains an incompressible surface. A 3-manifold is *atoroidal* if every incompressible torus in M is parallel to a component of ∂M. Jaco-Shalen [27] and Johannson [28] showed that each closed Haken 3-manifold contains a canonical family of disjoint incompressible tori $T_1, \dots, T_n$, called the *JSJ-decomposition* of M, such that each connected component of M split along $T_1, \dots, T_n$ is Seifert fibered or atoroidal. Thurston proved that the atoroidal pieces which are not Seifert are hyperbolic (see [37, 36, 29].) Note that the JSJ-decomposition may be empty. In this case, M is Seifert or hyperbolic. Therefore, the above-mentioned results prove Theorem 2.1 in the case where M is Haken.

We say that M is a *graph manifold* if it is a union of circle bundles glued along their boundaries. This notion was introduced by F. Waldhausen [48]. We collect some useful facts about graph manifolds in the next proposition:

Proposition 5.1. —

(i) *Any Seifert manifold is a graph manifold.*
(ii) *If M is a Haken manifold, then M is a graph manifold if and only if its JSJ-decomposition has only Seifert pieces.*
(iii) *If M is an irreducible graph manifold, then M is Seifert or contains an incompressible torus.*

We shall be concerned with the long time behavior of the Ricci flow, or weak solutions if Ricci flow is not defined for all time. More precisely, we look at the hyperbolically rescaled solution $\tilde{g}(t) = (4t)^{-1} g(t)$. Heuristically, one can expect $\tilde{g}(t))$ to converge to a hyperbolic metric on M if there exists one, and to collapse if M is a graph manifold. If M has a nonempty JSJ-decomposition with at least one hyperbolic piece, then it should split into a 'thick' part and a 'thin' part, where the thick part corresponds to the union of the hyperbolic pieces, and the 'thin' part corresponds to the union of the Seifert pieces, which is a graph manifold. These two parts should be separated by incompressible tori.

Remark 5.2. — It follows from a theorem of R. Myers [35] that any closed 3-manifold M contains a *hyperbolic knot*, i.e. an embedded circle whose complement is hyperbolic. In particular, M can be decomposed as the union of a hyperbolic manifold and a solid torus (which is a graph manifold). Thus when we work with the riemannian metrics given by weak solutions to the Ricci flow, it is crucial to show that the tori that appear as boundary components of the thick part are incompressible; otherwise, we would not obtain any further understanding of the topology of M.

Before closing this subsection, we define another classical notion in 3-manifold topology which is needed in the sequel: let X be a compact 3-manifold whose boundary is a (possibly empty) union of tori. We say that *M can be obtained from X by Dehn filling* if there exists an embedding $f : X \to M$ and a (possibly empty, possibly disconnected) 1-submanifold $L \subset M$ such that $M \setminus f(X)$ is a regular neighborhood of L. By convention, we allow L to be empty, so that the class of manifolds that can be obtained from M by Dehn filling contains M itself. This will be convenient later on.

If Y is an open 3-manifold diffeomorphic to $X \setminus \partial X$ and M can be obtained from X by Dehn filling, then we also say that M can be obtained from Y by Dehn filling. Thus the theorem of Myers quoted above implies that any closed 3-manifold can be obtained from a hyperbolic manifold by Dehn filling.

5.2. Long time behavior of weak solutions

Until the end of Section 5, M is a closed, irreducible 3-manifold whose fundamental group is infinite. By Lemma 2.2, this implies that M is aspherical.

Using the contrapositive of Theorem 4.2, it follows from our hypotheses that M does not admit locally canonical metrics. Hence Theorem 4.6 implies that for any initial condition, there exists a weak solution defined on any compact interval $[0,T]$. We need to strengthen this in two respects: first we need weak solutions defined on $[0,+\infty)$; second, we want them to satisfy a list of geometric properties relevant to the topological applications. In order to state those properties, we need some terminology.

Definition 5.3. — *Let $k > 0$ be a whole number. Let (M_n, g_n, x_n) be a sequence of pointed riemannian manifolds, and let $(M_\infty, g_\infty, x_\infty)$ be a pointed riemannian manifold. We shall say that (M_n, g_n, x_n)* converges *to $(M_\infty, g_\infty, x_\infty)$ in the $\mathcal{C}^k$-sense if there exists a sequence of numbers $\varepsilon_n > 0$ tending to zero, and a sequence of $\mathcal{C}^k$-diffeomorphisms φ_n from the metric ball $B(x_\infty, \varepsilon_n^{-1}) \subset M_\infty$ to the metric ball $B(x_n, \varepsilon_n^{-1}) \subset M_n$ such that $\varphi_n^*(g_n) - g_\infty$ has $\mathcal{C}^k$-norm less then ε_n everywhere. We say that the sequence* subconverges *if it has a convergent subsequence.*

Remark 5.4. — Note that the limit manifold M_∞ need not be homeomorphic to any of the M_n's. Typically, the M_n's are compact and M_∞ is noncompact. However, if M_∞ is compact, then for large n, the manifold M_n must be diffeomorphic to M_∞.

Definition 5.5. — *Let (X, g) be a riemannian* 3*-manifold. We say that a point $x \in X$ is ε*-thin with respect to g *if there exists a radius $\rho \in (0,1]$ such that the ball $B(x,\rho)$ has the following two properties: all sectional curvatures on this ball are bounded below by ρ^{-2}, and the volume of this ball is less than $\varepsilon\rho^3$. Otherwise, x is ε*-thick with respect to g*.*

We set

$$\hat{R}(g) := R_{\min}(g) \cdot \operatorname{vol}(g)^{2/3}.$$

This quantity has two important properties: it is scale-invariant, and it is nondecreasing along the Ricci flow on a closed manifold, as long as it remains nonpositive. This is also true for weak solutions by condition (ii) of Definition 4.5. Since our manifold M is aspherical, it does not admit any metric of positive scalar curvature [19, 42]. Hence $\hat{R}(g)$ is nonpositive for each metric g on M. As a consequence, we have:

Proposition 5.6. — *Let $\{g(t)\}$ be a weak solution on M. Then the function $t \mapsto \hat{R}(g(t))$ is nondecreasing.*

When H is a hyperbolic manifold, we let $\hat{R}(H)$ denote $\hat{R}(g_{\mathrm{hyp}})$, where g_{hyp} is the hyperbolic metric. Note that this number is equal to $-6 \cdot \operatorname{vol}(g_{\mathrm{hyp}})^{2/3}$, since g_{hyp} has constant scalar curvature equal to -6. Hence if H_1, H_2 are two hyperbolic manifolds, then $\operatorname{vol}(H_1) \leqslant \operatorname{vol}(H_2)$ if and only if $\hat{R}(H_1) \geqslant \hat{R}(H_2)$.

We are now in position to state the main result of this subsection (cf. [41, Sections 6 and 7]):

Theorem 5.7. — *Let M be a closed, irreducible, aspherical* 3*-manifold. For every riemannian metric g_0 on M, there exists a weak solution $g(t)$ defined on $[0,+\infty)$ with the following properties:*

(i) $g(0) = g_0$
(ii) *The volume of the hyperbolically rescaled metric $\tilde{g}(t)$ is bounded uniformly in t.*
(iii) *For every $\varepsilon > 0$ and every sequence $(x_n, t_n) \in M \times [0,+\infty)$, if t_n tends to $+\infty$ and x_n is ε-thick with respect to $\tilde{g}(t_n)$, then there exists a hyperbolic* 3*-manifold H and a basepoint $* \in H$ such that the sequence $(M, \tilde{g}(t_n), x_n)$ subconverges in the pointed $\mathcal{C}^2$ topology to $(H, g_{\mathrm{hyp}}, *)$. (Recall that for us, 'hyperbolic manifold' means complete and of bounded volume.)*
(iv) *For every sequence $t_n \to \infty$, the sequence $\tilde{g}(t_n)$ has controlled curvature in the sense of Perelman.*

'Controlled curvature in the sense of Perelman' is a technical property, which is weaker than a global two-sided curvature bound, but suffices for some limiting arguments. It will not be discussed here. See [4] for the definition.

Remark 5.8. — The assumption that M is irreducible is in fact redundant: since M is aspherical, any embedded 2-sphere in M bounds a homotopy 3-ball B; by the positive solution to the Poincaré Conjecture, B must be diffeomorphic to the 3-ball. We include this hypothesis to emphasize that various parts of the proof of Theorem 2.1 can be made independent from one another.

Remark 5.9. — Condition (iii) may be vacuous, i.e. there may exist no such sequence of ε-thick basepoints for any fixed ε. This happens for instance if $g(t)$ is a flat solution on a 3-torus.

5.3. Sequences of riemannian metrics on aspherical 3-manifolds

We begin with a direct corollary of Theorem 5.7:

COROLLARY 5.10. — *Let M be a closed, irreducible, aspherical 3-manifold. For every riemannian metric g_0 on M, there exists an infinite sequence of riemannian metrics $g_1, \dots, g_n, \dots$ with the following properties:*

(i) *The sequence $(\hat{R}(g_n))_{n \geqslant 0}$ is nondecreasing. In particular, it has a limit, which is greater than or equal to $\hat{R}(g_0)$.*
(ii) *The sequence $(\mathrm{vol}(g_n))_{n \geqslant 0}$ is bounded.*
(iii) *For every $\varepsilon > 0$ and every sequence $x_n \in M$, if x_n is ε-thick with respect to g_n, then there exists a hyperbolic 3-manifold H and a basepoint $* \in H$ such that the sequence (M, g_n, x_n) subconverges in the pointed $\mathcal{C}^2$ topology to $(H, g_{\mathrm{hyp}}, *)$.*
(iv) *The sequence g_n has controlled curvature in the sense of Perelman.*

Proof. — Applying Theorem 5.7, we obtain a weak solution $\{g(t)\}_{t \in [0,\infty)}$ with initial condition g_0. Set $g_n := \tilde{g}(t_n)$ for $n \geqslant 1$. Using Proposition 5.6 and the scale invariance of $\hat{R}$, we have

$$\hat{R}(g_0) \leqslant \hat{R}(g_1) \leqslant \cdots \leqslant \hat{R}(g_n) \leqslant \cdots .$$

This proves assertion (i). Assertions (ii), (iii) and (iv) follow from their counterparts in the conclusion of Theorem 5.7. □

The next task is to explore the topological consequences of the existence of a sequence of metrics satisfying the conclusion of Corollary 5.10. This leads to Proposition 5.11 below.

PROPOSITION 5.11. — *Let $g_0, g_1, \ldots$ be a sequence of riemannian metrics on M as in Corollary 5.10. Then one of the following conclusions hold:*

(i) *M is a graph manifold,*
(ii) *M is hyperbolic,*
(iii) *M contains an incompressible torus, or*
(iv) *There exists an open hyperbolic manifold H such that M can be obtained by Dehn filling on H, and $\hat{R}(H) \geqslant \hat{R}(g_0)$.*

Sketch of proof. — Up to extracting a subsequence, we distinguish several cases:

Case 1 ("collapsing case"). — There exists a sequence $\varepsilon_n \to 0$ such that every point of (M, g_n) is ε_n-thin. In this case, we prove that M is a graph manifold. Below we only give a quick sketch so that the reader can see the ideas involved. See [4] for the details.

Our approach is centered around the notion of *simplicial volume*, introduced by Gromov in [17]. The simplicial volume of a closed, orientable n-manifold X is defined by

$$\|X\| := \inf\{\sum_i |\alpha_i|, [X] = \sum_i \alpha_i \sigma_i\},$$

where $[X] \in H_n(X, \mathbf{R})$ is the fundamental class, the σ_i's are continuous maps of the standard n-simplex to X, and the α_i's are real numbers.

It is known [46] that if X is a Haken 3-manifold, then $\|X\|$ equals V_3 times the sum of the volumes of the hyperbolic pieces in the JSJ-decomposition of X, where V_3 is a universal constant. In particular, $\|X\| = 0$ if and only if X is a graph manifold.

Using the Cheeger-Gromov compactness theorem, we show that every point $x \in M$ has a neighborhood U_x whose geometry approximates a metric ball in a 3-manifold of nonnegative curvature. By Cheeger-Gromoll theory, there is a list of possible topologies for U_x; for instance, U_x might be a thickened torus $T^2 \times I$ or a solid torus $S^1 \times D^2$. All the U_x's have virtually abelian fundamental group. Using the hypothesis that M is aspherical, we show that there exists $x \in M$ such that the complement X of U_x in M is Haken. The next step is to prove that any closed 3-manifold obtained from X by Dehn filling has vanishing simplicial volume. Using Thurston's hyperbolic Dehn filling theorem and classical facts from 3-manifold topology, one deduces that X is a graph manifold, which implies that M is a graph manifold.

Case 2. — There exists $\varepsilon > 0$ and a sequence $x_n \in M$ such that x_n is ε-thick with respect to g_n.

Using Corollary 5.10 (iii), by further extracting a subsequence we can assume that (M, g_n, x_n) converges in the pointed $\mathcal{C}^2$-sense to some pointed hyperbolic manifold $(H^1, g_{\mathrm{hyp}}, *)$. If H^1 is closed, then M is diffeomorphic to H^1, hence M is hyperbolic. Thus the interesting case is when H^1 is noncompact. In this case, (M, g_n) contains for large n a submanifold $\bar{H}^1_n$ which is a large metric ball around x_n and diffeomorphic to a large ball in H. Thus each boundary component of $\bar{H}^1_n$ is a torus corresponding to some cusp cross-section in H.

Repeating this construction if necessary, we find a finite set of hyperbolic manifolds $H^1, \dots, H^p$ such that, for large n, the ε-thick part of (M, g_n) is covered by disjoint submanifolds $\bar{H}^1_n, \dots, \bar{H}^p_n$, which are approximated by large metric balls in the H^i's, and are bounded by approximately cuspidal tori. To prove that the construction stops for an integer p independent of n, we use the uniform bound on $\mathrm{vol}(g_n)$ and the Margulis Lemma.

All boundary components of the $\bar{H}^i_n$'s are tori. If one of these tori is incompressible in M, then we are done. Hence we assume that they are all compressible. Let X be a connected submanifold of M. We say that X is *abelian* if the image of the natural homomorphism $\pi_1 X \to \pi_1 M$ is abelian. This is the case, for instance, if some component of ∂X bounds a solid torus containing $\bar{H}^i_n$, or if $\bar{H}^i_n$ is contained in some topological 3-ball.

We have the following purely topological lemma:

Lemma 5.12. — *Let $X \subset M$ be a submanifold bounded by compressible tori. If X is non-abelian, then M can be obtained from X by Dehn filling.*

The proof then splits again into two cases: if all $\bar{H}^i_n$'s are abelian, then a refined version of the argument used for Case 1 shows that M is a graph manifold. If there exists i such that $\bar{H}^i_n$ is non-abelian, then using Lemma 5.12 applied to $X = \bar{H}^i_n$, we deduce that M can be obtained from $\bar{H}^i_n$ by Dehn filling. Since H^i is diffeomorphic to the interior of $\bar{H}^i_n$, M is also obtained from H^i by Dehn filling. From the monotonicity of $\hat{R}(g_n)$ and the fact that H^i is a pointed limit of (M, g_n), we get

$$\hat{R}(g_0) \leqslant \lim_{n\to\infty} \hat{R}(g_n) \leqslant \hat{R}(H^i).$$

This finishes our sketch of proof of Proposition 5.11. □

We continue the proof of Theorem 2.1. If conclusion (i), (ii), or (iii) of Proposition 5.11 holds, then by Proposition 5.1 the required topological conclusion has been reached. All that remains to do is to explain how the initial metric g_0 can be chosen so that conclusion (iv) is excluded.

At this point it is convenient to recall the following well-kown facts from hyperbolic geometry:

Theorem 5.13 (see [18]). — *The set of volumes of hyperbolic manifolds is well-ordered.*

Proposition 5.14 ([1]). — *Let H be an open hyperbolic 3-manifold and M be a closed 3-manifold obtained from H by Dehn filling. Then M carries a riemannian metric g_ε such that $\hat{R}(g_\varepsilon) > \hat{R}(H)$.*

Consider the collection of all hyperbolic manifolds H such that M can be obtained from H by Dehn filling. By the theorem of Myers quoted at the end of Subsection 5.1, this collection is never empty. Hence we can consider the infimum of the volumes of these manifolds, which we will denote by $V_0(M)$. By Theorem 5.13, this infimum is in fact a minimum.[(3)]

We are now ready for the last argument: let H_0 be a hyperbolic manifold realizing the minimum in $V_0(M)$. If M is not hyperbolic, then H_0 is open, and M is obtained from H_0 by Dehn filling. By Proposition 5.14, M admits a metric g_ε such that $\hat{R}(g_\varepsilon) > \hat{R}(H_0)$. Applying Theorem 5.7 with $g_0 = g_\varepsilon$ yields a sequence $g_1, g_2, \ldots$. If H is a hyperbolic manifold from which M can be obtained by Dehn filling, then by definition of $V_0(M)$, we have $\mathrm{vol}(H) \geqslant \mathrm{vol}(H_0)$. This implies that $\hat{R}(H) \leqslant \hat{R}(H_0) < \hat{R}(g_\varepsilon)$. Thus conclusion (iv) of Proposition 5.11 is excluded, and applying this proposition finishes the proof of Theorem 2.1 in the aspherical case.

Remark 5.15. — Since we allow ourselves to pass to subsequences, we do not prove that when M is hyperbolic, hyperbolically rescaled weak solutions starting from arbitrary metrics actually converge to the hyperbolic metric. This stronger statement is in fact true. Its proof requires additional arguments (see [41, 24, 30].)

Remark 5.16. — (*) The part of the proof that deals with compressible tori is inspired by [41, Section 8]. We have replaced Perelman's invariants $\hat{\lambda}$ and $\bar{V}$ by $\hat{R}$ and V_0 respectively. The idea to use $\hat{R}$ seems due to M. Anderson (see also [30, Section 93], where two versions of the argument along the lines suggested by Perelman are given.) The minimal volume $\bar{V}$ considered by Perelman and Kleiner-Lott is different from our V_0; for instance, $\bar{V}(M)$ is zero if M is a graph manifold, whereas $V_0(M)$ is always positive.

Our treatment of the collapsing case is completely different from the one hinted at by Perelman in [41, Section 7]. See [45] for another approach using Alexandrov space theory.

(3) Although it is not necessary for the proof, it is perhaps worth remarking that when M is hyperbolic, one can show that $V_0(M)$ is equal to the hyperbolic volume of M. This follows from the well-known fact that hyperbolic Dehn filling decreases volume, which is also conceptually connected to Proposition 5.14. Hence one can think of $V_0(M)$ as a replacement for the hyperbolic volume when the manifold is not hyperbolic.

6. More on weak solutions

The purpose of this section is to discuss some aspects of the proof of Theorem 4.6. We first give some background on the Ricci flow, then describe the metric surgery construction. In all of this section, M is a closed, irreducible, RP^2-free 3-manifold.

For an evolving metric $\{g(t)\}$ we denote the minimum (resp. maximum) of the scalar curvature of the time-t metric by $R_{\min}(t)$ (resp. $R_{\max}(t)$.)

6.1. Preliminaries

A riemannian metric on M is *normalized* if it has sectional curvature between 1 and -1, and the volume of any ball of unit radius is greater than or equal to half the volume of a Euclidean ball of unit radius. Since M is compact, any metric can be normalized by scaling. Moreover, the property of being locally canonical for a metric is scale invariant. Hence it suffices to prove Theorem 4.6 for normalized initial conditions.

Let $I \subset [0, +\infty)$ be an interval. Following the terminology of [34], we say that an evolving metric $\{g(t)\}_{t\in I}$ has *curvature pinched toward positive* if for every $(x, t) \in M \times I$ the following two conditions hold:

$$R(x,t) \geqslant -\frac{6}{4t+1} \tag{6.1}$$

(6.2)

$$R(x,t) \geqslant 2X(x,t)(\log X(x,t) + \log(1+t) - 3) \text{ whenever } X(x,t) > 0,$$

where $X(x, t)$ is the opposite of the lowest eigenvalue of the curvature operator of $g(t)$ at x.

Below we give a few basic properties of the Ricci flow on M.

Proposition 6.1 (Long time existence). — *Let K be a positive number. If g_0 is a metric satisfying $|\mathrm{Rm}| \leqslant K$, then the Ricci flow solution with initial condition g_0 exists on $[0, 2^{-4}K^{-1}]$ and satisfies $|\mathrm{Rm}| \leqslant 2K$ on this interval.*

Proposition 6.2 (Curvature estimates). — *Let $I \subset [0, +\infty)$ be an interval containing 0. Let $\{g(t)\}_{t\in I}$ be a Ricci flow solution with normalized initial condition. Then $g(t)$ has curvature pinched toward positive.*

Seeking to prove Theorem 4.6, we fix a number $T > 0$ and a normalized initial condition g_0. It follows from Proposition 6.2 that $R_{\min}(t)$ is uniformly bounded from below. Moreover, if we have a solution defined on some interval $[0, t_0]$ and such that $R_{\max}(t)$ is bounded from above on $[0, t_0]$, then by Proposition 6.2, the norm of the curvature tensor is also bounded above on this interval. Hence by Proposition 6.1, the solution can be prolonged on a slightly larger interval $[0, t_0 + \alpha]$.

As a result, Theorem 4.6 is only difficult to prove if the Ricci flow solution with initial condition g_0 is defined on a maximal interval $[0, T')$ with $T' < T$ and $R_{\max}(t)$ is unbounded as $t \to T'$. That is why we need a good description of the regions of M where the scalar curvature becomes large. Such a description is provided by the following theorem of Perelman (cf. [39, Theorem 12.1].)

THEOREM 6.3. — *For every $\varepsilon > 0$ and every $T > 0$, there exists $r = r(\varepsilon, T) > 0$ such that if $\{g(t)\}$ is a Ricci flow solution on M defined on $[0, T]$ with normalized initial condition, then for all $(x_0, t_0) \in M \times [0, T]$ such that $R(x_0, t_0) \geqslant r^{-2}$, the point (x_0, t_0) has an ε-canonical neighborhood in $(M, \{g(t)\})$.*

The definition of ε-canonical neighborhood is a bit technical, so it will not be given here. It suffices to know two things: first, if (x_0, t_0) has an ε-canonical neighborhood in $(M, \{g(t)\})$, then x has a weak ε-canonical neighborhood in $(M, g(t_0))$; second, it implies the differential inequality

$$\frac{\partial R}{\partial t} < C|R|^2, \tag{6.3}$$

where C is a constant depending only on ε. (In the sequel, ε will be fixed, so C will be universal.)

DEFINITION 6.4. — *An evolving metric $\{g(t)\}_{t \in I}$ satisfies the* Canonical Neighborhoods *property with parameter r (henceforth abbreviated as $(CN)_r$) if for all $(x, t) \in M \times [0, T]$ such that $R(x, t) \geqslant r^{-2}$, the point (x, t) has an ε-canonical neighborhood in $(M, g(t))$.*

6.2. Surgery on δ-necks

We now describe the surgery procedure. We fix a small number $\varepsilon > 0$. This number should satisfy various conditions, e.g. it should be less than 10^{-2} so that Theorem 4.2 holds.

The two main parameters that govern the surgery procedure are called r and δ. The parameter r is related to the curvature scale above which points have canonical neighborhoods; it has the dimension of length, and is smaller than the number $r(\varepsilon, T)$ given by Theorem 6.3. The parameter δ describes the precision of the surgery; it is dimensionless, and much smaller than ε. Assume for the moment that r and δ have been fixed.

The following technical result is adapted from Lemma 4.3 of [41].

Theorem 6.5 (Existence of cutoff parameters). — *There exist positive numbers $h < \delta r$ and $D \geqslant 10$ depending only on δ and r, such that if $(M, \{g(t)\})$ is a weak solution satisfying $(CN)_r$, t is a time in the domain of definition, and x, y, z are points of M such that $R(x,t) \leqslant 2r^{-2}$, $R(y,t) = h^{-2}$, $R(z,t) \geqslant Dh^{-2}$, and y lies on a minimizing $g(t)$-geodesic connecting x to z, then y is the center of a δ-neck.*

We now carry out the construction outlined in Section 4, setting the curvature threshold to $2Dh^{-2}$. Precisely, this means the following: we let $t_0 \leqslant T$ be the first time where $R_{\max}(t)$ reaches $\Theta := 2Dh^{-2}$ (if there is no such time, then there is nothing to prove.) If all points of $(M, g(t_0))$ have scalar curvature greater than r^{-2}, then by Theorem 6.3 they all have canonical neighborhoods, and we are done. From now on, we assume that it is not the case. We partition M into three subsets $\mathcal{R}, \mathcal{O}, \mathcal{G}$ defined as follows:

$$\begin{aligned}
\mathcal{R} &:= \{z \in M \mid Dh^{-2} \leqslant R(z,t_0)\} \\
\mathcal{O} &:= \{y \in M \mid 2r^{-2} < R(y,t_0) < Dh^{-2}\} \\
\mathcal{G} &:= \{x \in M \mid R(x,t_0) \leqslant 2r^{-2}\}.
\end{aligned}$$

Intuitively, $\mathcal{R}$ is the set of 'red' points, where the scalar curvature is huge, and a singularity is threatening to appear. The purpose of the surgery operation is to remove those points. The set $\mathcal{G}$ consists of 'green' points, where R cannot be large, and $\mathcal{O}$ is the set of 'orange' points, where R may be large, but not that much.

By assumption, $\mathcal{G}$ and $\mathcal{R}$ are not empty. Let x be a point of $\mathcal{G}$, z be a point of $\mathcal{R}$, and γ be a minimizing geodesic connecting x to z. Then by the intermediate value theorem, there exists a point $y \in \gamma$ whose scalar curvature is exactly h^{-2}. Applying Theorem 6.5, we obtain a δ-neck centered on y. One can show that this can be repeated finitely many times to yield a finite collection of disjoint δ-necks $N_1, \dots N_p$ whose union separates $\mathcal{R}$ from $\mathcal{G}$.

Let us denote by $X_1, \dots, X_q, X_{q+1}, \dots, X_{q+s}$ the connected components of $M \setminus \bigcup_i N_i$, where $X_j \subset \mathcal{G} \cup \mathcal{O}$ for $j \leqslant q$ and $X_j \subset \mathcal{O} \cup \mathcal{R}$ for $j > q$. By a relative version of Theorem 4.2, the high curvature components $X_{q+1}, \dots, X_{q+s}$, which are covered by canonical neighborhoods, are diffeomorphic to B^3 or $S^2 \times I$. Since M is irreducible, a straightforward topological argument shows that there is a not-so-large-curvature component X_{j_0}, $j_0 \leqslant q$, such that $M \setminus X_{j_0}$ is a finite, disjoint union of topological 3-balls in M, and each point of ∂X_{j_0} is the center of a δ-neck.

In other words, $M \setminus X_{j_0}$ is covered by a union $\Sigma(t_0)$ of δ-caps. The surgery operation consists in replacing those caps by special caps, called 'almost standard caps'. The precise definition is too technical to be discussed here. The main points are that the post-surgery metric $g_+(t_0)$ has curvature pinched toward positive, and at all points of $\Sigma(t_0)$ its scalar curvature is comparable to h^{-2}. As a result, $R_{\max}(g_+(t_0))$ is bounded above by $\Theta/2$. After this, we restart the Ricci flow with new initial condition $g_+(t_0)$.

As long as condition $(VC)_r$ is satisfied, we can iterate this construction. Since points of scalar curvature between r^{-2} and Θ have canonical neighborhoods, they satisfy inequality (6.3). Together with the fact that surgery makes $R_{\max}$ drop by at least half its value, this ensures that there is a definite lower bound for the time span between two consecutive surgeries. Hence the iteration of this process produces a weak solution.

To prove Theorem 4.6, we need to show that the parameters r and δ can be chosen so that the construction can indeed be iterated until it produces a weak solution defined on $[0, T]$ or a locally canonical metric. Theorem 6.3 does not suffice for this; instead, we need to generalize it to a special class of weak solutions. This is done in a complex limiting argument involving such notions as κ-solutions or κ-noncollapsing, which I will not attempt to explain here.

Remark 6.6. — (*) Since we work on a compact time interval rather than $[0, +\infty)$, the parameters r and δ are fixed rather than time-dependent as in [41]. This simplification was observed by Perelman in [40]. To prove Theorem 5.7 however, we need to work with time-dependent parameters, which creates an additional layer of complexity.

Remark 6.7. — (*) As we already noticed, the main difference between our construction and Perelman's is that we do surgery before the singularity appears, rather than at the singular time. As a consequence, we do not have to discuss horns, capped horns and double horns. Our solution for a given initial condition may be very different from Perelman's: at one extreme, Ricci flow may be defined for all time: then Perelman's construction produces the Ricci flow solution, while ours may lead to surgery if the curvature reaches the threshold. Neither construction leads to a *canonical* 'Ricci flow through singularities'.

Another consequence of this choice is that we do not explicitly address the question of formation of singularities in the Ricci flow. The reader should be aware, however, that the difficulties we have to face and the way they are overcome (e.g. blow-up arguments) are essentially the same.

Remark 6.8. — (*) Our argument to rule out accumulation of surgeries is different from Perelman's, and does not use the volume. This is crucial for the generalization to infinite volume solutions discussed below.

7. Extension to non-compact 3-manifolds

Let M be a possibly noncompact 3-manifold without boundary. A riemannian metric g on M has *bounded geometry* if it has bounded sectional curvature and injectivity radius bounded away from zero. An evolving metric $\{g(t)\}_{t\in I}$ is *complete* (resp. *has bounded geometry*, resp. *has bounded sectional curvature*) if for each $t \in I$ the metric $g(t)$ has the corresponding property.

The results of Section 4 can be extended to this context. For simplicity we give the existence result for *irreducible* open 3-manifolds (note that such a manifold is automatically RP^2-free.)

Theorem 7.1. — *Let M be an open, irreducible 3-manifold not diffeomorphic to $\mathbf{R}^3$. Let g_0 be a complete riemannian metric on M which has bounded geometry. Then there exists a complete weak solution $\{g(t)\}_{t\in[0,\infty)}$ of bounded geometry on M with initial condition $g(0) = g_0$.*

Theorem 7.1 is proved using the construction described in Section 6 and the following version of Theorem 4.2 for open 3-manifolds:

Theorem 7.2. — *Let (M, g) be an open riemannian 3-manifold. If g is locally canonical, then M is diffeomorphic to $\mathbf{R}^3$ or $S^2 \times \mathbf{R}$. In particular, if M is irreducible, then M is diffeomorphic to $\mathbf{R}^3$.*

Sketch of proof of Theorem 7.1. — One defines parameters r, δ, h, D, Θ as in the compact case. If the maximum of the scalar curvature reaches the threshold Θ for some time t_0, there is a similar decomposition of M into three subsets $\mathcal{R}$, $\mathcal{O}$, and $\mathcal{G}$, which may have infinitely many components. By Theorem 7.2, $g(t_0)$ cannot be locally canonical, so $\mathcal{G}$ is nonempty. Using an generalization of Theorem 6.5, one finds a (possibly infinite) set of spheres in $\mathcal{O}$ which are middle spheres of δ-necks and separate $\mathcal{R}$ from $\mathcal{G}$. Using elementary 3-manifold topology, one deduces from irreducibility of M that there is a noncompact submanifold $X \subset M$ containing no point of $\mathcal{R}$, and whose complement is a (possibly infinite) set of disjoint 3-balls. Then one performs geometric surgery on each of these 3-balls in order to decrease the maximal scalar curvature. The rest of the argument goes through without major changes. □

Theorem 7.1 has an application to positive scalar curvature, which we now describe. Let us denote by $R_{\min}(g)$ the infimum of the scalar curvature of a metric g (which may or may not be attained.) Following [19], we say that g has *uniformly positive scalar curvature* if $R_{\min}(g) > 0$.

THEOREM 7.3. — *Let M be an open, irreducible 3-manifold which carries a complete metric g_0 of bounded geometry and uniformly positive scalar curvature. Then M is diffeomorphic to $\mathbf{R}^3$.*

Proof. — If M is not diffeomorphic to $\mathbf{R}^3$, then we can apply Theorem 7.1 to get a complete weak solution $\{g(t)\}$ of bounded geometry with initial condition g_0, defined on $[0, +\infty)$. A maximum principle argument (cf. [8, §2.1]) together with condition (ii)(a) of the definition of weak solution shows that

$$R_{\min}(t) \geqslant \frac{R_{\min}(0)}{1 - 2tR_{\min}(0)/3}$$

if $g(t)$ is a complete weak solution with bounded sectional curvature. This gives an upper bound on the lifetime of such a solution, hence a contradiction. □

Theorem 7.3 is a special case of a more general result. In order to state it, we need a definition of connected sum allowing infinitely many summands.

If $\mathcal{X}$ is a class of closed 3-manifolds, we say that a 3-manifold M is a *connected sum* of members of $\mathcal{X}$ if there exists a locally finite simplicial tree T and a map $v \mapsto X_v$ which associates to each vertex of T a manifold in $\mathcal{X}$, such that by removing from each X_v as many 3-balls as vertices incident to v and gluing the thus punctured X_v's to each other along the edges of T, one obtains a 3-manifold diffeomorphic to M.

For instance, both $\mathbf{R}^3$ and $S^2 \times \mathbf{R}$ can be viewed as connected sums of S^3's, the tree being a half-line in the former case, and a line in the latter.

It is easy to see that if one takes $\mathcal{X}$ to be the class containing spherical 3-manifolds and manifolds diffeomorphic to $S^2 \times S^1$, then any connected sum of members of $\mathcal{X}$ admits a complete riemannian metric of uniformly positive scalar curvature. The following result is a partial converse to this:

THEOREM 7.4. — *Let M be a 3-manifold which carries a complete metric g_0 of bounded geometry and uniformly positive scalar curvature. Then M is a connected sum of spherical manifolds and copies of $S^2 \times S^1$.*

To prove Theorem 7.4, one needs to work with a more general notion of weak solution, where the manifold is allowed to change with time. The surgery process may disconnect the manifold, and break it into possibly infinite connected sums. However, the essence of the proof is already contained in Theorem 7.1.

BIBLIOGRAPHY

1. Michael Anderson, *Scalar curvature and the existence of geometric structures on 3-manifolds. I*, J. Reine Angew. Math. **553** (2002), 125–182.
2. Sigurd Angenent and Dan Knopf, *An example of neckpinching for Ricci flow on* S^{n+1}, Math. Res. Lett. **11** (2004), no. 4, 493–518.
3. Laurent Bessières, Gérard Besson, Michel Boileau, Sylvain Maillot, and Joan Porti, *Géométrisation des variétés de dimension 3*, In preparation.
4. ______, *Weak collapsing and geometrisation of aspherical 3-manifolds*, ArXiv:0706.2065, Jun 2007.
5. Gérard Besson, *Preuve de la conjecture de Poincaré en déformant la métrique par la courbure de Ricci (d'après Perelman)*, Séminaire Bourbaki 2004-2005, Astérisque, vol. 307, Société Mathématique de France, Paris, France, 2006, pp. 309–348.
6. Michel Boileau, *Uniformisation en dimension trois*, Astérisque **266** (2000), Exp. No. 855, 4, 137–174, Séminaire Bourbaki, Vol. 1998/99.
7. Michel Boileau, Sylvain Maillot, and Joan Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses, vol. 15, Société Mathématique de France, Paris, 2003.
8. Huai-Dong Cao and Xi-Ping Zhu, *A Complete Proof of the Poincaré and Geometrization Conjectures - application of the Hamilton-Perelman theory of the Ricci flow*, Asian Journal of Mathematics **10** (2006), no. 2, 165–492, Revised version available on the arXiv.
9. Andrew Casson and Douglas Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math. **118** (1994), 441–456.
10. Jeff Cheeger and Michael Gromov, *Collapsing riemannian manifolds while keeping their curvature bounded I*, J. Differential Geometry **23** (1986), 309–346.
11. ______, *Collapsing riemannian manifolds while keeping their curvature bounded II*, J. Differential Geometry **32** (1990), 269–298.
12. Bennett Chow and Dan Knopf, *The Ricci flow : an introduction*, Mathematical surveys and monographs, vol. 110, A.M.S., 2004.
13. Tobias H. Colding and William P. Minicozzi, II, *Estimates for the extinction time for the Ricci flow on certain 3-manifolds and a question of Perelman*, J. Amer. Math. Soc. **18** (2005), no. 3, 561–569 (electronic).
14. ______, *Width and finite extinction time of ricci flow*, ArXiv:0707.0108, Jul 2007.
15. Kenji Fukaya, *Collapsing Riemannian manifolds and its applications*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990) (Tokyo), Math. Soc. Japan, 1991, pp. 491–500.
16. David Gabai, *Convergence groups are Fuchsian groups*, Annals of Math. **136** (1992), 447–510.
17. M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5–99 (1983). MR MR686042 (84h:53053)

18. Michael Gromov, *Hyperbolic manifolds (according to Thurston and Jørgensen)*, Bourbaki Seminar, Vol. 1979/80, Lecture Notes in Math., vol. 842, Springer, Berlin, 1981, pp. 40–53.

19. Michael Gromov and Blaine Lawson, Jr, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 83–196 (1984).

20. Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

21. ———, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), no. 2, 153–179.

22. ———, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), Internat. Press, Cambridge, MA, 1995, pp. 7–136.

23. ———, *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. **5** (1997), no. 1, 1–92.

24. ———, *Non-singular solutions of the Ricci flow on three-manifolds*, Comm. Anal. Geom. **7** (1999), no. 4, 695–729.

25. John Hempel, *3-manifolds*, Annals of mathematics studies, vol. 086, Princeton University Press, Princeton, 1976.

26. William Jaco, *Lectures on three-manifold topology*, CBMS Lecture Notes, vol. 43, American Mathematical Society, Providence, Rhode Island, USA, 1980.

27. William H. Jaco and Peter B. Shalen, *Seifert fibered spaces in 3-manifolds*, Memoirs of the American Mathematical Society, no. 220, American Mathematical Society, Providence, Rhode Island, USA, September 1979.

28. Klaus Johannson, *Homotopy equivalences of 3-manifolds with boundary*, Lecture Notes in Mathematics, vol. 761, Springer, Berlin, 1979.

29. Michael Kapovich, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics, vol. 183, Birkhäuser Boston Inc., Boston, MA, 2001.

30. Bruce Kleiner and John Lott, *Notes on Perelman's papers*, ArXiv: math.DG/0605667, May 2006.

31. John Lott, *On the long-time behavior of type-III Ricci flow solutions*, Math. Ann. **339** (2007), no. 3, 627–666.

32. Sylvain Maillot, *Open 3-manifolds whose fundamental groups have infinite center, and a torus theorem for 3-orbifolds*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4595–4638 (electronic). MR MR1990764 (2004m:57036)

33. Geoffrey Mess, *The Seifert conjecture and groups which are coarse quasiisometric to planes*, Preprint.

34. John Morgan and Gang Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, vol. 3, American Mathematical Society, Providence, RI, 2007.

35. Robert Myers, *Simple knots in compact, orientable* 3*-manifolds*, Trans. Amer. Math. Soc. **273** (1982), no. 1, 75–91.

36. Jean-Pierre Otal, *Thurston's hyperbolization of Haken manifolds*, Surveys in differential geometry, Vol. III (Cambridge, MA, 1996), Int. Press, Boston, MA, 1998, pp. 77–194.

37. ———, *The hyperbolization theorem for fibered 3-manifolds*, SMF/AMS Texts and Monographs, vol. 7, American Mathematical Society, Providence, RI, 2001, Translated from the 1996 French original by Leslie D. Kay.

38. P. Pansu, *Effondrement des variétés riemanniennes, d'après J. Cheeger et M. Gromov*, Astérisque **121-122** (1985), 63–82, Seminar Bourbaki, Vol. 1983/84.

39. Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications*, ArXiv : math.DG/0211159, Nov 2002.

40. ———, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, ArXiv : math.DG/0307245, Jul 2003.

41. ———, *Ricci flow with surgery on three-manifolds*, ArXiv : math.DG/0303109, Mar 2003.

42. Richard Schoen and Shing-Tung Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) **110** (1979), no. 1, 127–142.

43. Peter Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

44. ———, *There are no fake Seifert fibered spaces with infinite π_1*, Annals of Math. **117** (1983), 35–70.

45. Takashi Shioya and Takao Yamaguchi, *Volume collapsed three-manifolds with a lower curvature bound*, Math. Ann. **333** (2005), no. 1, 131–155.

46. Teruhiko Soma, *The Gromov invariant of links*, Invent. Math. **64** (1981), no. 3, 445–454.

47. Pekka Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. **391** (1988), 1–54.

48. Friedhelm Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, II*, Invent. Math. 3 (1967), 308–333; ibid. **4** (1967), 87–117.

Sylvain Maillot
Université Louis Pasteur
Institut de Recherche Mathématique Avancée
7 rue René Descartes
67084 Strasbourg cedex (France)
maillot@math.u-strasbg.fr