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ON RANK ONE SYMMETRIC SPACE

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Abstract

In this paper we survey some recent results on rank one symmetric space.

Résumé

Dans ce papier, nous survolons quelques résultats récents sur l'espace symétrique de rang un.

1. Introduction

Symmetric space has been at the center of research interest more than several decades. It is tied up with Riemannian geometry, semi-simple Lie group theory, ergodic theory, 3-dimensional topology, number theory and other branches of mathematics. It is a Riemannian space which has both rigidity and rich deformation, and available research tools are elegantly intertwined each other. It has a rigidity nature whose prototypical theorems are by Mostow and Margulis. So the fundamental group of a compact locally symmetric manifold completely determines the geometry in dimension ≥ 3. It also has an entropy result whose best known theorem is by Besson, Courtois and Gallot [2]. But if we change the realm to infinite volume locally symmetric manifolds, it has a rich deformation theory. Convex cocompact hyperbolic 3-manifold is the best example to study. Hyperbolic geometry is also a basic tool to classify 3-dimensional topological manifolds after Thurston. After Perelman's work, 3-dimensional topology is closely related to partial differential equation theory and analysis on manifolds. In this short note, we survey some recent results related to this rigidity-flexibility phenomenon of symmetric space, specially of rank 1. We begin with 3-dimensional hyperbolic geometry which provides a rich deformation theory. Its deformation theory is pioneered by Alfors, Bers, Kra, Sullivan, Thurston and many others. Recently many open conjectures are settled down and we present one of this conjecture in

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this paper, namely convergence criterion for freely decomposable Kleinian groups. Next we look at rigidity side of rank one space in the language of length spectrum. Finally we prove a local rigidity of a lattice in a bigger symmetric space.

2. Hyperbolic 3-manifold

According to Thurston's geometrization program, every 3-manifold can be decomposed into smaller pieces so that each piece admits one of eight geometries. Hyperbolic geometry is dominant and most important in many aspects. When proposing this geometrization program, Thurston himself proved a remarkable theorem, which is a Haken hyperbolization theorem. During his proof of this theorem, mapping torus was the most important and hardest case to deal with. To prove this case, he used so-called a *double-limit theorem*. But his technique in this case heavily depends on the fact that the boundary of a manifold is incompressible. So this technique fails in compression body. But he proposed a similar theorem which is valid even in a manifold with compressible boundary. Recently the author with two others settled down this conjecture and in fact we gave a stronger version of Thurston's conjecture [8]. The precise statement is

THEOREM 1. — *Let M be a compact irreducible atoroidal 3-manifold with a compressible boundary, and $\rho_0 : \pi_1(M) \to PSL(2, \mathbb{C})$ a geometrically finite representation which uniformises M. Let (m_i) be a sequence in the Teichmüller space $\mathcal{T}(\partial M)$ which converges in the Thurston compactification to a projective measured lamination $[\lambda]$ which is doubly incompressible. Let $q : \mathcal{T}(\partial M) \to QH(\rho_0)$ be the Alfors-Bers map, and suppose that ρ_i is a sequence of discrete faithful representations corresponding to $q(m_i)$. Then passing to a subsequence, ρ_i converges algebraically.*

A 3-manifold is irreducible if every embedded two sphere bounds a 3-ball, and is atoroidal if every torus is boundary parallel. A boundary of a 3-manifold is compressible if it's fundamental group does not inject into the fundamental group of the 3-manifold, and equivalently by loop theorem (or Dehn's lemma) it is compressible if the 3-manifold has a properly embedded compressing disc whose boundary lies on that boundary. For a given H^3/Γ, one can form the smallest convex set whose boundary is the limit set of Γ, which is called the convex hull of Γ. The quotient of the convex hull by Γ is called the convex core of Γ. This definition extends to any negatively curved manifolds. If the convex core has a finite volume, the manifold is called *geometrically finite*. If there is no cusps, it is called *convex cocompact*. A measured lamination λ is doubly incompressible if there is a positive number η so that for any compressing disc, essential annulus or Möbius band E, $i(\lambda, \partial E) > \eta$. Recall also that Thurston compactified the Teichmüller space by attaching the space of projective measured laminations as a boundary. An Alfors-Bers map $q : \mathcal{T}(\partial M) \to QH(\rho_0)$ is the one which associates a convex cocompact hyperbolic metric in the interior of M to its hyperbolic metric on boundary surface at infinity.

This map is a covering map in general. $QH(\rho_0)$ is the space of quasi-conformal, equivalently quasi-isometric deformations of a given metric ρ_0.

The idea of a proof goes as follows. Suppose the sequence ρ_n diverges. Then it converges to an isometric action on $\mathbb{R}$-tree in the sense of Gromov-Morgan-Shalen. An $\mathbb{R}$-tree is a geodesic metric space such that every pair of points has a unique geodesic connecting them. After a hard working, one can create a sequence of annuli A_i in M so that the Housdorff limit of ∂A_i does not intersect λ transversely. But this will contradict the property of λ being doubly incompressible. Using this theorem together with *ending lamination conjecture*, one can deduce *Ber's density conjecture*. Namely any finitely generated Kleinian group is an algebraic limit of a sequence of geometrically finite Kleinian groups.

3. Length spectrum rigidity

When a locally symmetric manifold is of infinite volume, Mostow rigidity does not hold, so to get a rigidity result one has to put more conditions. The most popular one is the marked length spectrum. Under this assumption, it is known that any Zariski dense subgroup of semi-simple Lie group of noncompact type is completely determined by its marked length spectrum [9, 6]. But it is widely believed that the unmarked length spectrum will determine the group up to finite possibilities. We call the set of lengths of closed geodesics (or the set of translation lengths of elements in a group) *length spectrum* or *length spectral set* of a manifold (or a group).

McKean [12] used the Selberg's trace formula to show that there is only finitely many hyperbolic metrics on a closed surface with a given spectral set. Later Osgood, Phillips and Sarnak [13] showed the compactness of isospectral metrics on a closed surface. Much later Brooks, Perry and Peterson [3] showed the same result for closed 3-manifolds near a metric of constant curvature.

When the manifold is of infinite volume, the following is known [10].

THEOREM 2. — *Let Λ be the length spectrum of a convex cocompact hyperbolic 3-manifold with incompressible boundary. Then there is only finitely many (up to isometry) hyperbolic manifolds homotopy equivalent to M which have the same length spectrum Λ.*

The proof is a combination of the result in [10] and *Mumford compactness theorem* on Teichmüller space. But this theorem is generalised to arbitrary rank 1 manifold with mild assumptions [5].

THEOREM 3. — *Suppose M is a convex cocompact real hyperbolic 3-manifold which is not a handle body. Then the set of convex cocompact real hyperbolic 3-manifolds homotopy equivalent to M with the same spectral set as M is finite up to isometry. For real hyperbolic manifold of dimension $n \geq 4$, the set of convex* *cocompact real hyperbolic manifold of dimension n which are homotopy equivalent to M with a fixed length spectrum is finite. For other rank 1 symmetric space, assume that the critical exponent of the Poincaré series is greater than or equal to the half of the Hausdorff dimension of the boundary of the symmetric space. Then the set of convex cocompact rank 1 metrics on a fixed manifold is finite.*

The idea of a proof goes as follows. One uses geometric limit argument in the sense of Gromov. First use that the bottom spectrum of Laplacian is bounded above by Cheeger constant, and then use the fact that there is a relation between the bottom spectrum and the Hausdorff dimension of the limit set. Then one can conclude that there is a uniform bound for the volume of the boundary of convex core, which has a fixed topology, using a version of Mostow rigidity for variable negative curvature [1]. Putting all these facts together, one gets that the volume of the convex core has a uniform upper bound for a fixed spectral set. Also for a fixed spectral set, one knows the uniform lower bound for injectivity radius of convex core, and so the upper bound for the diameter of the convex core. Then using a geometric limit method, together with the result in [9] about a finite determining set, one finishes the proof.

For hyperbolic 3-manifold, one used an assumption that the bottom spectrum is not 1, which occurs when the manifold is a handle body. For higher dimensional real hyperbolic space, one can use the argument of Sullivan [14] to derive the finiteness. One suspect that there is a different method to show the finiteness of isospectrality for higher rank symmetric spaces.

4. Local rigidity

Even though Mostow rigidity rules for lattices, one can think of the following situation. Let Γ be a lattice in a symmetric space S. Consider a bigger symmetric space S' which contains S as a totally geodesic manifold. Γ naturally acts on S'. Is it possible to deform Γ in $\mathrm{Iso}(S')$? In some cases, the answer is yes. There are examples of lattices in $PSO(3,1)$ which can be deformed in $PSO(k,1)$ for $k > 3$. But in general this is not always true. For example a uniform lattice in $PSU(n,1)$ cannot be deformed in $PSU(m,1), m > n$ due to K. Corlette [4]. Following the arguments used in [7], one can see that the fundamental group of a convex cocompact rank one locally symmetric manifold of dimension > 3 is isomorphic to the one of the CW complex whose complexity is controlled by the volume of the manifold. So for a fixed upper bound for the volume of the convex core of the manifold, there are only finitely many possible fundamental groups. Then suppose that a lattice in a symmetric space S can be deformed to a convex cocompact discrete group in S' where S is totally geodesically embedded. One can perturb so little that the volume of the convex core is bounded above by a uniform constant. Then by the observation above, this is possible only for finitely many lattices. This shows that most of cases, we cannot deform a lattice in a bigger symmetric space.

In this note we give another example of local rigidity [11].

THEOREM 4. — *Let* Γ *be a uniform lattice in* $PSO(4,1)$ *which can be regarded as a discrete group in* $PSp(n,1), n > 1$ *in a canonical way by identifying* $H^4_{\mathbb{R}}$ *with a quaternionic line. Then* Γ *cannot be locally deformed in* $PSp(n,1)$.

The idea of a proof is to use Raghunathan-Matsushima-Murakami result. We can push the result a litter further to deal with non-uniform lattice with an assumption.

THEOREM 5. — *Under the same assumption with a non-uniform lattice, there is no local deformation preserving parabolicity.*

This fact has to do with L^2-norm of forms. Note that Raghunathan-Matsushima-Murakami result holds only for finite L^2-norm forms. We strongly believe that in our case, we do not need preserving parabolicity. The same statement holds for uniform lattice in $SU(1,1)$ and in $SO(3,1)$ considering $SU(1,1) \subset Sp(1,1) \subset Sp(n,1)$ and $SO(3,1) \subset SO(4,1) \subset Sp(1,1) \subset Sp(n,1)$.

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