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MILNOR-WOOD INEQUALITY FOR CRYSTALLOGRAPHIC GROUPS

Hiroyuki MINAKAWA

0. Introduction

Let H^2 be the hyperbolic plane and $\mathrm{Isom}^+ H^2$ the isometry group of H^2. A 2-dimensional crystallographic group Γ is a cocompact discrete subgroup of $\mathrm{Isom}^+ H^2$. As an abstract group, Γ is isomorphic to a unique group of the form

$$\begin{gathered}\Gamma(g; p_1, \cdots, p_n) = \langle a_1, b_1, \cdots, a_n, b_n, c_1, \cdots, c_n | \\ c_i^{p_i} = 1 \ (i = 1, \cdots, n), \\ c_1 \cdots c_n [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle\end{gathered}$$

with $g \geq 0, p_i \geq 2$ and $\chi(\Gamma(g; p_1, \cdots, p_n)) < 0$. Here $\chi(\Gamma(g : p_1, \cdots, p_n)) = 2 - 2g - \sum_{i=1}^{n} (p_i - 1)/p_i$ is the rational Euler characteristic of the group $\Gamma(g; p_1, \cdots, p_n)$.

Let G^r be the group of all orientation preserving diffeomorphisms of class C^r ($r = 0, 1, \cdots, \infty$). For any homomorphism $\phi : \Gamma \to G^r$, Γ acts on the trivial S^1 bundle $H^2 \times S^1$ through ϕ. So we can construct a foliated Seifert bundle $E_\phi = H^2 \times S^1/\Gamma \to H^2/\Gamma = \Sigma_g (g = \text{genus of } \Gamma)$. We define the Euler number $eu(\phi)$ of ϕ by

$$\begin{aligned} eu(\phi) &= \text{the Euler number of Seifert bundle } E_\phi \to \Sigma_g \\ &= eu(E_\phi \to \Sigma_g). \end{aligned}$$

If Γ is a surface group, then we have the Milnor-Wood inequality

$$|eu(\phi)| \leq |\chi(\Sigma)| = |\chi(\Gamma)|.$$

Moreover, if $\phi_i : \Gamma \to G^0 (i = 1, 2)$ both have the maximal Euler number $eu(\phi_1) = eu(\phi_2) = \pm\chi(\Gamma)$, then ϕ_1 is semi-conjugate to ϕ_2.

In this paper, we shall consider a generalization of the Milnor-Wood inequality for homomorphisms from crystallographic groups to G^0, and we also prove that there exists a semi-conjugacy phenomenone in the case that the homomorphism has the maximal Euler number.

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1. Homological definition of Euler number

In this section, we give a homological definition of Euler number $eu(\phi)$ first.

Let $\Gamma = \Gamma(g; p_1, \cdots, p_n)$ be a crystallographic group. Γ contains a finite index subgroup $\Gamma_{g'}$ which is isomorphic to the fundamental group of a closed surface $\Sigma_{g'}$. So we have that the inclusion $i : \Gamma_{g'} \to \Gamma$ induces an isomorphism

$$i_* : H_2(\Gamma_{g'}; Q) \to H_2(\Gamma; Q).$$

Since given presentation of $\Gamma_{g'}$ determins an orientation of the closed surface $\Sigma_{g'}$, then there exists the fundamental class $[\Gamma_{g'}] \in H_2(\Gamma_{g'}; Z) \cong H_2(\Sigma_{g'})$. We use the notation $[\Gamma_{g'}]_Q$ which is the image of $[\Gamma_{g'}]$ by Bockstein homomorphism

$$H_2(\Gamma_{g'}; Z) \to H_2(\Gamma_{g'}; Q).$$

Now we define the fundamental class $[\Gamma]$ of Γ by

$$[\Gamma] = i_*[\Gamma_{g'}]_Q/\mathrm{index}(\Gamma; \Gamma_{g'}).$$

We can check easily that this definition does not depend on the choice of the finite index subgroup $\Sigma_{g'}$.

2. Cohomological definition of the Euler number

Given a surface group Γ_g and a homomorphism $\phi : \Gamma_g \to G^0$, Euler number $eu(\phi)$ is equal to the pairing

$$eu(\phi) = \langle \phi^* e, [\Gamma_g] \rangle.$$

Here, $e \in H^2(G^0; Z)$ denotes the universal Euler class. The symbol e_Q denotes the rational universal Euler class which is the image of e by Bockstein homomorphism $H^2(G^0; Z) \to H^2(G^0; Q)$.

PROPOSITION 2.1. — *For any homomorphism $\phi : \Gamma \to G^0$, we have the formula*

$$eu(\phi) = \langle \phi^* e_Q, [\Gamma] \rangle.$$

In order to prove thes proposition, we need the following lemma.

LEMMA 2.2. — *Let $\pi_i : M_i \to \Sigma_i (i = 1, 2)$ be Seifert fibrations. Assume that there exist maps $\tilde{h} : M_1 \to M_2$ and $h : \Sigma_1 \to \Sigma_2$ such that $\pi_2 \circ \tilde{h} = h \circ \pi_1$, degree$(h) = k$ and degree$(\tilde{h}|_{\text{reguler fiber}}) = l$. Then we have $e(M_1 \to \Sigma_1) = (k/l) eu(M_2 \to \Sigma_2)$.*

Proof of Proposition 2.1 We take a finite index subgoup $\Gamma_{g'}$ of Γ which is isomorphic to $\pi_1(\Sigma_{g'})$. We put that $k = \mathrm{index}(\Gamma; \Gamma_{g'})$. So there exist continuous maps

$\tilde{h} : E_{\phi\circ i} \to E_\phi$ and $h : \Sigma_{g'} \to \Sigma_g$ such that $\pi_\phi \circ \tilde{h} = h \circ \pi_{\phi\circ i}$, degree$(h) = k$ and degree$(\tilde{h}|_{\text{regular fiber}}) = 1$. By using the lemma above, we have

$$\begin{aligned} eu(\phi) &= eu(E_\phi \to \Sigma_g) \\ &= eu(E_{\phi\circ i} \to \Sigma_{g'})/k \\ &= eu(\phi \circ i)/k \\ &= \langle (\phi \circ i)^* e, [\Sigma_{g'}] \rangle / k \\ &= \langle (\phi \circ i)^* e_Q, [\Sigma_{g'}]_Q \rangle / k \\ &= \langle \phi^* e_Q, i_* [\Sigma_{g'}]_Q / k \rangle \\ &= \langle \phi^* e_Q, [\Gamma] \rangle . \end{aligned}$$

□

The same technique as in the proof of Proposition 2.1 gives us a generalization of the Milnor-Wood inequality for homomorphisms from crystallographic groups to G^0.

THEOREM 2.3. — *Let Γ be a crystallographic group. For any homomorphism $\phi : \Gamma \to G^0$, we have the following inequality*

$$|eu(\phi)| \leq |\chi(\Gamma)|.$$

Proof. We use the same notations as in the proof of Proposition 2.1. Then we have

$$|eu(\phi)| = |eu(\phi \circ i)/k| \leq |\chi(\Gamma_{g'})|/k = |\chi(\Gamma)|.$$

The last equality follows from the definition of the rational Euler characteristic $\chi(\Gamma)$ (see [8]). □

3. Semi-conjugacy in maximal Euler numbers

Let Γ be a crystallographic group and $T_1 H^2$ a unit tagent bundle of the hyperbolic plane H^2. Γ acts on $T_1 H^2$, since Γ acts on H^2 isometrically. So we can construct a Seifert bundle $E(\Gamma) = T_1 H^2/\Gamma \to H^2/\Gamma = \Sigma_g$ whose tatal holonomy homomorphism is the identity map $\phi_\Gamma : \Gamma \to \Gamma \subset PSL(2, R)$. We know that $eu(\phi_\Gamma) = \chi(\Gamma)$. The following theorem is a generalization of a theorem of S.Matsumoto to crystallographic groups. In [6], he proved this theorem for surface groups.

THEOREM 3.1. — *Let Γ be as above. For given homomorphism $\phi : \Gamma \to G^0$, there exist a continuous degree one map $h : S^1 \to S^1$ such that*

$$\phi_\Gamma(\gamma) \circ h = h \circ \phi(\gamma)$$

for any $\gamma \in \Gamma$.

By [5], it suffices to show that

$$\rho(\phi_\Gamma(\gamma)) = \rho(\phi(\gamma))$$

for any $\gamma \in A$ which is a generating system of Γ. Here $\rho(f) \in S^1$ is rotation number of $f \in G^0$. In order to show this, we need the following formula which is called Milnor's algorism.

LEMMA 3.2. — *For any homomorphism $\phi : \Gamma(g; p_1, \cdots, p_n) \to G^0$ we can calculate the Euler number $eu(\phi)$ as follows. We choose any lifts $\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}, \widetilde{\phi(c_i)} \in G^0$. Then, the number*

$$\tilde{\rho}([\widetilde{\phi(a_1)}, \widetilde{\phi(b_1)}] \circ \cdots \circ [\widetilde{\phi(a_g)}, \widetilde{\phi(b_g)}] \circ \widetilde{\phi(c_1)} \circ \cdots \circ \widetilde{\phi(c_n)}) + \sum_{i=1}^{n} \rho(\widetilde{\phi(c_i)})$$

does not depend on the choice of lifts. This number is equal to $eu(\phi)$.

Where, $\tilde{\rho}(\tilde{f})$ is the traslation number of $\tilde{f}$. We can prove the following lemma by Lemma 3.2 with [1], [4] and [7].

LEMMA 3.3. — *For any homomorphism $\phi : \Gamma(g; p_1, \cdots, p_n) \to G^0$, we have that*

$$\rho(\phi(\gamma)) = \begin{cases} 0 & \text{if } \gamma = a_1, \cdots, a_g, b_1, \cdots, b_g \\ [1/p_i] & \text{if } \gamma = c_i (i = 1, \cdots, n) \end{cases}$$

if $eu(\phi) = \chi(\Gamma)$.

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