Séminaire de Théorie spectrale et géométrie

V. F. Lazutkin

Recent results on the separatrix splitting for the standard map

Séminaire de Théorie spectrale et géométrie, tome 11 (1992-1993), p. 157-165
<http://www.numdam.org/item?id=TSG_1992-1993__11__157_0>

Séminaire de théorie spectrale et géométrie
GRENOBLE
1992–1993 (157–165)

RECENT RESULTS ON THE SEPARATRIX SPLITTING FOR THE STANDARD MAP

V.F. LAZUTKIN

ABSTRACT. **This is a short survey of recent works concerning separatrix splittings for the standard map. Different approaches give an asymptotic expansion for the intersection angle of the separatrices of the hyperbolic fixed point at the origin.**

§1. THE FORMULA FOR THE SPLITTING

The followimg selfmap, of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, the so called *standard map*, attracted the attention of mathematicians and physicists in last decades [C],[G]:

$$\begin{aligned} SM\colon (X,Y) &\longmapsto (X_1,Y_1) \\ X_1 &= X + Y_1\,, \quad \bmod 2\pi \\ Y_1 &= Y + \varepsilon \sin X \quad \bmod 2\pi\,. \end{aligned} \tag{1.1}$$

This map is a good model for the study of the coexistence of regular and chaotic behaviour of trajectories in Hamiltonian systems with two degrees of freedom. If $\varepsilon = 0$, the transformation (1.1) is integrable: the lines $y =$ const are invariant circles,the map being a rotation on each such circle. What happens if ε is small positive? The point $(0,0)$ becomes a fixed hyperbolic point with the largest eigenvalue of the linear part

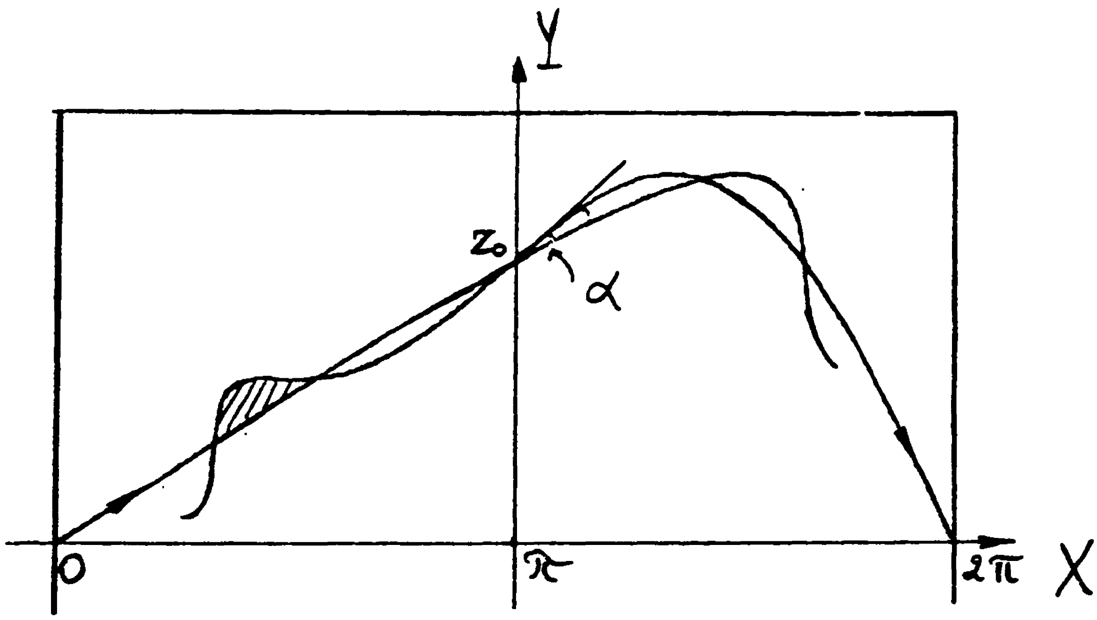
$$\lambda = 1 + \frac{\varepsilon}{2} + \sqrt{\varepsilon + \frac{\varepsilon^2}{4}}\,. \tag{1.2}$$

The stable, W^s, and the unstable, W^u, manifolds of this fixed point have many intersection, the *homoclinic* points. Let z_0 be the first point of their intersection which occurs at the line $x = \pi$ (the first means in the sence of linear ordering along W^u starting with z_0, one can prove the existence of such a point). Denote by α the angle under which W^u and W^s intersect at z_0. The following asymptotic formula is true when $\varepsilon \to 0$:

$$\alpha \sim \frac{\pi\omega_0}{\varepsilon} e^{-\pi^2/\sqrt{\varepsilon}} \tag{1.3}$$

where [S]:

$$\omega_0 = 1118.82770594090077842\ldots. \tag{1.4}$$



INTERSECTION OF THE STABLE AND UNSTABLE MANIFOLDS

We see that this angle is nonzero. This results in a very complicated behaviour of $W^{s,u}$, creation of so called ***stochastic layer***. It was A.Poincaré who first described such a phenomenon in [PO].

Before describing the way how this (and a more refined,(1.12)) result can be obtained, let us dwell on the numerical calculation of the prefactor ω_0. We give here a formal recipe.

Consider the following difference equation:

$$u(x+1)+u(x-1)-2u(x)=\exp\big(u(x)\big) \tag{1.5}$$

and try to find a formal solution to (1.5) in the form:

$$u_0^-(x)=-\log\frac{x^2}{2}+\sum_{k=1}^{\infty}p_{0k}x^{-2k}\,. \tag{1.6}$$

where p_{0k} are real numbers. Substituting (1.6) into (1.5) and equalizing the coefficients at powers of x^2 at both sides of (1.5), one gets $p_{01}=-\frac{1}{4}$, $p_{02}=\frac{91}{864}$, $p_{03}=-\frac{319}{2880},\dots,$ and a recurrent equation for p_{0k} from which they can be found uniquely (see [GLST] Appendix A1). The following formula expresses the preexponent factor in (1.3) in terms of the constructed sequence:

$$\omega_0=\lim_{k\to\infty}24\pi^3(-1)^k\frac{(2\pi)^{2k}}{(2k+1)!}\cdot p_{0k}\,. \tag{1.7}$$

In fact a more refined formula than (1.3) is valid. To write it down we need some preliminaries. First, it is more useful a small parameter

$$h=\log\lambda=\log\left(1+\frac{\varepsilon}{2}+\sqrt{\varepsilon+\frac{\varepsilon^2}{4}}\right)\approx\sqrt{\varepsilon}\,. \tag{1.8}$$

instead of ε. Second, it is preferable to calculate a so called ***homoclinic invariant*** ω instead of the intersection angle α. To define this quantity, let us write the equation of the "upper" half of W^u in a parametric form:

$$X = X^-(t,\varepsilon), \quad Y = Y^-(t,\varepsilon), \quad t \in \mathbb{R}, \tag{1.9}$$

and choose the parameter t so that

(1) the standard map restricted onto the upper half of W^u reads in terms of that parameter as a shift $t \mapsto t+h$ where h is given by (1.8);
(2) there is a kind of a regular behaviour of $X^-(t,\varepsilon)$ as $t \to -\infty$, namely

$$X^-(t,\varepsilon) = Ce^t + \mathcal{O}(e^{2t}) \tag{1.10}$$

if $t \to -\infty$, minus infinity corresponding to the fixed point $(0,0)$.

These requirements define the parametrisation (1.9) uniquely up to adding a constant to the variable t. So the following tangent vector to W^u at z_0 is defined uniquely:

$$\mathbf{e}^u(z_0) = \left(\frac{dX^-(t,\varepsilon)}{dt}, \frac{dY^-(t,\varepsilon)}{dt}\right)\Bigg|_{\left(X^-(t,\varepsilon),Y^-(t,\varepsilon)\right)=z_0}$$

Analogously, one can define a natural tangent vector $\mathbf{e}^s(z_0)$ to W^s at z_0. Let $\Omega = dX \wedge dY$ be the standard symplectic structure on the torus, which SM preserves. We define the homoclinic invariant of the intersection of W^u and W^s at z_0 as

$$\omega = \Omega(\mathbf{e}^u, \mathbf{e}^s).$$

This quantity is the same for all points of the homoclinic trajectory passing through z_0. I does not change under canonical (=area and orientation preserving) transformations. One can easily deduce that

$$\omega = \|\mathbf{e}^u\| \cdot \|\mathbf{e}^s\| \cdot \sin\alpha \tag{1.11}$$

but the multipliers in the right-hand side of (1.11) depend on the choice of a Riemannian metric.

A more refined formula for the separatrix splitting reads:

$$\omega = \frac{\pi}{h^2} e^{-\frac{\pi^2}{h}} \left[\sum_{n=0}^{\infty} \frac{h^{2n}}{n!}\omega_n\right]. \tag{1.12}$$

The sum in the right-hand side of (1.12) is an asymptotic one. This means that if one cuts this sum to a finite one, the error will be of the order of the first thrown up term. The coefficients in (1.12) can be calculated with the use of computers, we shall discuss this later. The value of ω_0 is given in (1.4), some other values are:

$$\begin{aligned} \omega_1 &= 18.59891\ldots, \\ \omega_2 &= -4.34411\ldots, \\ \omega_3 &= -4.1829\ldots, \\ \omega_4 &= -4.88\ldots \end{aligned} \tag{1.13}$$

The formula (1.3) was firstly obtained in [L1], see also [L2],[LST]. The relation (1.7) was discovered by Yu.B.Suris [Su]. Carles Simó used (1.7) to calculate with high presision the value (1.4). Other maps and systrens were considered in [FS],[GLT],[HMS]. An exponetialy small estimate for general analytic area-preserving map was firstly obtained by A.Neishtadt [N]. A refined formula (1.12) was derived in [GLS].

§2. Behaviour of the unstable manifold in the complex domain

To obtain an asymptotics to the homoclinic invariant we investigate the analytic extension of W^u into the complex domain. It is sufficient to explore the first component $X^-(t,\varepsilon)$ of (1.9) because $Y^-(t,\varepsilon) = X^-(t,\varepsilon) - X^-(t-h,\varepsilon)$. This first component can be found from the equation

$$\Delta_h^2 X^- = \varepsilon \sin X^- , \tag{2.1}$$

where

$$\Delta_h^2 f(t) = f(t+h) + f(t-h) - 2f(t), \tag{2.2}$$

under the condition (1.10). To fix the constant C of (1.10), we require that

$$X(0) = \pi . \tag{2.3}$$

The condition (2.3) fixes the choice of the variable t along the unstable manifold. We fixed it so that $t = 0$ corresponds to the intersection point z_0. One can prove that there exists a unique solution to the equation (2.1) completed with the conditions (1.10) and (2.3), and this solution is an entire function of the variable t. This enables us to consider a continuation of $X^-(t,\varepsilon)$ into the complex domain of the variable t. Since ε (or equivalently h) is a small parameter it is natural to seek an aymptotics for $X^-(t,\varepsilon)$ in the form

$$X^-(t,\varepsilon) = \sum_{n=0}^{\infty} \frac{h^{2n}}{n!} X_n(t), \tag{2.4}$$

where $X_n(t)$ do not depend on ε. It was proven in [GLS] that $X_n(t)$ has a form

$$X_0(t) = 4 \arctan \mathrm{e}^t , \tag{2.5}$$

and for $n \geq 1$

$$X_n(t) = \sum_{k=1}^{n} \frac{a_{nk}}{(\cosh t)^{2k}} \cdot \sinh t \tag{2.6}$$

where a_{nk} are real numbers. In particular,

$$\begin{aligned} X_1(t) &= \frac{1}{4} \cdot \frac{\sinh t}{(\cosh t)^2} , \\ X_2(t) &= -\frac{41}{864} \cdot \frac{\sinh t}{(\cosh t)^2} + \frac{91}{432} \cdot \frac{\sinh t}{(\cosh t)^4} . \end{aligned} \tag{2.7}$$

The constructed functions have singularities at the points $t = \pm i\frac{\pi}{2} + i2k\pi$, $k \in \mathbb{Z}$, while $X^-(t,\varepsilon)$ is an entire function of the variable t. This shows that the series (2.4) cannot approximate $X^-(t,\varepsilon)$ in a neighborhood of the mentioned singularities.So it is important to know where (2.4) does approximate our function. Since $X^-(t,\varepsilon)$ is real on the real axis an so are the coefficients in the right hand side of (2.4), it is sufficient to consider them for $\operatorname{Im} t \geq 0$, as it follows from the symmetry reasons. One can prove that the series (2.4) is asymtotic in any domain of the form

$$\mathcal{D} = \left\{ t \in \mathbb{C}\colon 0 \leq \operatorname{Im} t \leq \frac{\pi}{2},\ \operatorname{Re} t \leq h,\ \arg\left(t - i\frac{\pi}{2}\right) \leq -\delta_0 < 0 \right\} .$$

One founds that, in addition to the boundary conditions (1.10), the formal series in the right-hand side of (2.4) satisfies the analogous condition at $+\infty$. Therefore the constructed series approximates the stable separatrix as well as the unstable one. One should replace the domain $\mathcal{D}$ by another domain changing the sign of t for the Proposition 4.5 to be true in the case of the stable separatrix. It follows that the constructed series cannot reveal the splitting of the separatrices.

In order to find the splitting one should investigate the behaviour of the function $X^-(t,\varepsilon)$ in a neighborhood of the singularity of $X_n(t)$ that is nearest to the real axis. To do that, let us pass to another function by the change:

$$X^-\left(i\frac{\pi}{2} + hx, \varepsilon\right) = -i \log\frac{\varepsilon}{2} + iU^-(x,\varepsilon). \tag{2.8}$$

The equation (2.1) converts into

$$\Delta^2 U^- = \mathrm{e}^{U^-} - \frac{\varepsilon^2}{4}\exp(-U^-), \tag{2.9}$$

where

$$\Delta^2 f(x) = f(x+1) + f(x-1) - 2f(x).$$

We expand U^- in the series of the form

$$U^-(x,\varepsilon) = \sum_{n=0}^{\infty} \frac{h^{2n}}{n!} u_n^-(x) \tag{2.10}$$

where $u_n^-(x)$ depend only on x. It is not difficult to write down the equations which $u_n^-(x)$ obey:

$$\Delta^2 u_0^- = \mathrm{e}^{u_0^-} , \tag{2.11}$$

$$\Delta^2 u_n^- - \mathrm{e}^{u_0^-} u_n^- = F_n\left(u_1^-, u_2^-, \ldots, u_{n-1}^-\right). \tag{2.12}$$

One can prove that there exists a unique sequence $u_n^-(x)$, $n = 0, 1, 2, \ldots$, of entire functions which satisfies the equations (2.11), (2.12) and such that the expansion in the right-hand side of (2.10) is asymptotic in any domain of the form

$$\tilde{\mathcal{D}}_1 = \left\{ x \in \mathbb{C}\colon |\mathrm{Re}\, x| \le 1\,, \quad -\frac{1}{\sqrt{h}} \le \mathrm{Im}\, x \le 0\,, \quad \pi \le \arg x \le 2\pi - \delta_0 \right\}$$

In any sector $\delta_0 \ge \arg x \ge 2\pi - \delta_0$, $\delta_0 > 0$, the functions $u_n^-(x)$ have the following asymptotic expansions:

$$u_0^-(x) = -\log\frac{x^2}{2} + \sum_{k=1}^{\infty} p_{0k} x^{-2k}\,, \tag{2.13}$$

$$u_n^-(x) = \sum_{k=-n}^{\infty} p_{nk} x^{-2k}\,. \tag{2.14}$$

The branch of logarithm in (2.13) is chosen so that the function (2.13) is real on the real axis of the variable x. The coefficients p_{nk} in (2.13) and (2.14) are real numbers.

§3. Comparing the stable and unstable manifold

The equation of the stable manifold, W^s, can be written in the analogous parametric form:

$$X = X^+(t, \varepsilon)\,, \quad Y = Y^+(t, \varepsilon)\,, \quad t \in \mathbb{R}\,, \tag{3.1}$$

and after rescaling

$$X^+\left(i\frac{\pi}{2} + hx, \varepsilon\right) = -i\log\frac{\varepsilon}{2} + iU^+(x, \varepsilon)\,. \tag{3.2}$$

One finds that

$$U^+(x, \varepsilon) = -2\pi i + U^-(-x, \varepsilon)\,, \tag{3.3}$$

and $U^+(x, \varepsilon)$ satisfies the same equation (2.9) as $U^-(x, \varepsilon)$, but with regularity condition at $x \to +\infty$. The difference

$$W(x, \varepsilon) = U^+(x, \varepsilon) - U^-(x, \varepsilon)$$

is small on the real line of the variable $t = i\frac{\pi}{2} + hx$. So, writing

$$W(x, \varepsilon) = \mathrm{e}^{-i2\pi x}\hat{W}(x, \varepsilon) + \mathcal{O}\left(\mathrm{e}^{-i4\pi x}\right)\,, \tag{3.4}$$

we expect that $\hat{W}(x, \varepsilon)$ is a solution to the linearized equation:

$$\Delta^2\Phi = \left(\mathrm{e}^{U^-} + \frac{\varepsilon^2}{4}\mathrm{e}^{-U^-}\right)\Phi\,. \tag{3.5}$$

One can construct two linearly independent formal solutions, Φ_1 and Φ_2 of (3.5), the first one being

$$\Phi_1(x,\varepsilon) = \frac{d}{dx}U^-(x,\varepsilon) \sim \sum_{n=0}^{\infty} \frac{h^{2n}}{n!}\varphi_{1,n}(x) \tag{3.6}$$

where

$$\varphi_{1,n}(x) = \frac{d}{dx}u_n^-(x),$$

and the second one, Φ_2, which satisfies the normalizing condition

$$\Phi_1\overline{\Delta}\Phi_2 - \Phi_2\overline{\Delta}\Phi_1 = 1, \tag{3.7}$$

where

$$\overline{\Delta}f(x) = f(x) - f(x-1).$$

One can find Φ_2 so that

$$\Phi_2(x,\varepsilon) \sim \sum_{n=0}^{\infty} \frac{h^{2n}}{n!}\varphi_{2,n}(x), \tag{3.8}$$

where $\varphi_{2,n}(x)$ are entire functions which are real on the real axis of the variable x and admit the following asymptotic expansion

$$\varphi_{2,n}(x) \sim \sum_{k=-n-1}^{\infty} \frac{\varphi_{nk}}{x^{2k}}, \qquad \varphi_{nk} \in \mathbb{R}, \tag{3.9}$$

in any sector $\delta \leq \arg x \leq 2\pi - \delta, \quad \delta > 0$.

The solution $\hat{W}(x,\varepsilon)$ can be expanded into a sum of two linearly independent solutions:

$$\hat{W}(x,\varepsilon) = \mu(h)\Phi_1(x,\varepsilon) + \theta(h)\Phi_2(x,\varepsilon) \tag{3.10}$$

where the coefficients are formal series:

$$\theta(h) \sim \sum_{n=0}^{\infty} \frac{h^{2n}}{n!}\theta_n, \tag{3.11}$$

$$\mu(h) \sim \sum_{n=0}^{\infty} \frac{h^{2n}}{n!}\mu_n, \tag{3.12}$$

θ_n, μ_n being numbers.

It was shown in [GLS] that

$$\theta_n = -i\omega_n \tag{3.13}$$

where ω_n are the coefficients of the splitting formula (1.12).

One of the ways of computing ω_n is to use Fourier analysis to select the first harmonic in (3.4) and expand it into the basis Φ_1, Φ_2. Using this method we calculated in [GLS] the values (1.13) and first digits in (1.4).

§4. Calculating the splitting via resurgence

There exists another way of calculation of the coefficients in the expansion (1.12) which results in (1.7) and similar formulae for other coefficients. That way comes from Jean Écalle's theory of résurgence (see for example [CNP] and references therein).

A cruicial step is to apply the Borel transform to the series (2.13) and (2.14). We define formally the Borel transform of U in two steps:

(1) omit the logarithm term in (2.13) and the terms with $k \le 0$ in (2.14);
(2) substitute $\frac{t^{2k-1}}{(2k-1)!}$ in the place of x^{-2k}.

In such a way we obtain the formal series

$$\mathbf{U}(t) = \sum_{n=0}^{\infty} \frac{h^{2n}}{n!} \mathbf{u}_n(t) \tag{4.1}$$

where

$$\mathbf{u}_n(t) = \sum_{k=1}^{\infty} \frac{p_{nk}}{(2k-1)!} t^{2k}. \tag{4.2}$$

It was shown in [L3] that $\mathbf{u}_0(t)$ is analytic in a neighbourhood of the origin of the variable t, and the nearest singularities are situated at the points $t = \pm 2\pi i$. Obviously the same is true for $\mathbf{u}_n(t)$ with $n \ge 1$. Note that $\mathbf{u}_n(t)$ are real on the real axis.

We conjecture the follwing analytical behaviour of $\mathbf{u}_n(t)$ at the point $t = 2\pi i$:

$$\begin{aligned} \mathbf{u}_n(t) = &\sum_{s=1}^{2n+3} \frac{A_{n,-s}(i)^{s+1}}{(t-2\pi i)^s} \\ &+ \log(t-2\pi i)\left[\sum_{s=0}^{\infty} A_{ns}(i)^{s+1}(t-2\pi i)^s\right] \\ &+ \text{holomorphic function}. \end{aligned} \tag{4.3}$$

Here A_{ns} are real numbers. One can determine the coefficients A_{ns} from the asymptotics of the coefficients p_{nk} for large values of k. $\hat{W}$ of the equation (3.5) defined by the relation (3.4), coincides with the residium of $\mathbf{U}$ at $2\pi i$ times $2\pi i e^{2\pi i x}$. Indeed, $U^{\pm}(x)$ can be reperesented as a Laplace transform of $\mathbf{U}$, the integral running over the path going from the origin to $\pm\infty$ on the real axis. Their difference, W, is the integral over the real axis, and it can be calculated by moving the contour upwards.

As it was already mentioned, the coefficients $A_{n,s}$ can be calculated from the asymtotic behaviour of $p_{n,k}$. On the other hand, by use of the residims, the formers are linked with $\hat{W}$, and hence with the coefficients μ and θ in (3.10). This link gives a way to calculate the coefficients ω_n in the splitting formula (1.12). The simpliest formula is (1.7).

We refer the reader to [L3] and [LS] for more details.

REFERENCES

[C] B. V. CHIRIKOV, *A universal instability of many-dimensional oscillator systems*, Physics Reports **52** (1979), 263–279.

[CNP] B. CANDELPERGER, J.-C. NOSMAS, F. PHAM, *Premiers pas en calcul étranger*, Ann. Inst. Fourier, Grenoble **43**, no.1 (1993), 201–224.

[FS] E. FONTICH, C. SIMÓ, *The splitting of separatrices for analytic diffeomorphisms*, Ergod. Th. Dynam. Sys. **10** (1990), 295–318.

[GLS] V. G.GELFREICH , V.F. LAZUTKIN, N. V. SVANIDZE, *A refined formula for the separatrix splitting for the standard map*, Physica D (to appear).

[GLST] V. G.GELFREICH , V.F. LAZUTKIN, C. SIMÓ , M.B. TABANOV, *Fern-like structures in the wild set of the standard and semistandard maps in* $\mathbb{C}^2$, International Journal of Bifurcation and Chaos **2**, no.2 (1992), 353–370.

[GLT] V. G. GELFREICH, V. F. LAZUTKIN , M. B. TABANOV, *Exponentially small splitting in hamiltonian systems*, Chaos **1**, no.2 (1991), 137–142.

[G] J. M. GREENE, *A method for determining a stochastic transition* , J. Math. Phys. **20** (1979), 1183–1201.

[GP] J. M. GREENE, I. C. PERCIVAL, *Hamiltonian maps in the complex plane*, Physica D **3** (1981), 530–548.

[HMS] P. HOLMES, J.MARSDEN, J. SCHEURLE, *Exponentially Small Splitting of Separatrices with Applications to KAM Theory and degenerate Bifurcations*, Contemporary Mathematics **81** (1988), 213–244.

[L1] V. F. LAZUTKIN, *Splitting of separatrices for the Chirikov's standard map.*, Preprint (1984), VINITI 6372/84.

[L2] V. F. LAZUTKIN, *Exponential splitting of separatrices and an analytical integral for the semistandard map*, Preprint no.7, Université Paris VII (1991), 53 p.

[L3] V.F. LAZUTKIN, *Resurgent approach to the separatrices splitting*, EQUADIFF 91, Iternational Conference on Differential Equations, Barcelona, Spain, 26-31 August 1991, vol.1 (C.Perelló, C.Simó, J.Solà-Morales, eds.), World Scientific, Singapore-New Jersey-London-Hong Kong, 1993, pp. 163–176.

[LS] V. F. LAZUTKIN , C. SIMÓ, *Separatrix splitting for the standard map via resurgence*, in preparation.

[LST] V. F. LAZUTKIN , I. G. SCHACHMANSKI M. B. , TABANOV, *Splitting of separatrices for standard and semistandard mappings*, Physica D **40** (1989), 235–248.

[N] A.I. NEISHTADT, *On the accuracy of the perturbation theory for systems with one rapid variable*, Prikladnaya Matematika i Mekhanika **45** (1981), 80–87. (Russian)

[Pe] I. C. PERCIVAL, *Chaotic boundary of a Hamiltonian map*, Physica D **6** (1982), 67–77.

[Po] H. POINCARÉ, , *Les Méthodes Nouvelles de la Méchanique Céleste*,Vols 1–3, Gauthier–Villars, Paris, 1892, reprinted by Dover, New York (1957).

[Si] C. SIMÓ, private communication.

[Su] Yu. B. SURIS, private communication.

[R] J. RIORDAN, *An introduction to combinatorial analysis*, Princeton University Press, Princeton, Guilford, 1980.

V.F. LAZUTKIN
St PETERSBURG STATE UNIVERSITY
Dept of mathematical physics
Ulyanov Str. 1, Kor. 1,
Petrodvorets
ST PETERSBURG 198904
(Russie)